d. Numerical Examples

This two-dimensional algorithm is available as an educational module from the LCSE. This module, the Wave Table, (figure 7.9) is divided into three windows, a control window, a viewing window, and a graphical window allowing the student to examine the wave amplitudes along a horizontal slice. The simulations which follow were performed using this module.

Reflection and Refraction

The laws of reflection and refraction are easily demonstrated. First create a long narrow "wall" with a fairly high index of refraction of 1.60. Rotate the wall by 45°, placing it in the middle of the work areas as shown in figure 7.10. The program has only one mode: "plane wave," which is an endless series of sine waves travelling from left to right. When a wave reaches the surface, it is split into a reflected wave and a transmitted wave, as expected from the one-dimensional string program described earlier. However, the additional degrees of freedom allowed in multi-dimensions requires that the vector direction of the wave be followed. The law of reflection is

$$\theta' = \theta, \tag{7.40}$$

that is, the angle of reflection equals the angle of incidence. Here, the the 45° surface causes waves which were initially travelling left to right to travel from bottom to top after reflection.

The transmitted wave enters a medium in which the speed of light is slower than in the external medium (here assume vacuum). That portion of the wave falls behind the rest of the wave. It must continue at this slower speed until it reaches the far edge of the wall, where it is split into a reflected and transmitted wave. The transmitted wave resumes the previous wave speed, but is now trailing the position it would otherwise have. It is this difference in wave speeds which leads directly to the law of refraction:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2, \tag{7.41}$$

where n is the index of refraction, and θ is the angle the direction of wave motion makes with the surface normal, resp. The subscripts refer to the two different media.

It should be noted that *wave fronts* are shown rather than the *rays* commonly encountered in textbooks. The rays are constructed from joining the instantaneous direction vectors along the wave fronts as shown in figure 7.11. This leads to the common optical design technique of ray tracing, but tends to somewhat obscure the underlying physical nature of refraction, which is due to the differing wave speeds in the two media. The many internal reflections and transmissions can often create a confusing picture, in reality as well as in this simulation. The ping mode may be used to follow the progress of a single wave.

Lenses

This difference in wave speeds handily explains how lenses affect light. Construct a thick convex (converging) lens as shown in figure 7.12 and start a wave train. The center of the wave front enters the lens first and slows. Additional portions of the wave form enter the lens at progressively later times, figure 7.13a. This results in a curved wave front (figure 7.13b) which converges to a point. The opposite happens with a concave (diverging) lens shown in figure 7.14. The outer portions of the wave front enter the lens before the central portions (figure 7.15a), resulting in a diviering wave front (figure 7.15b).
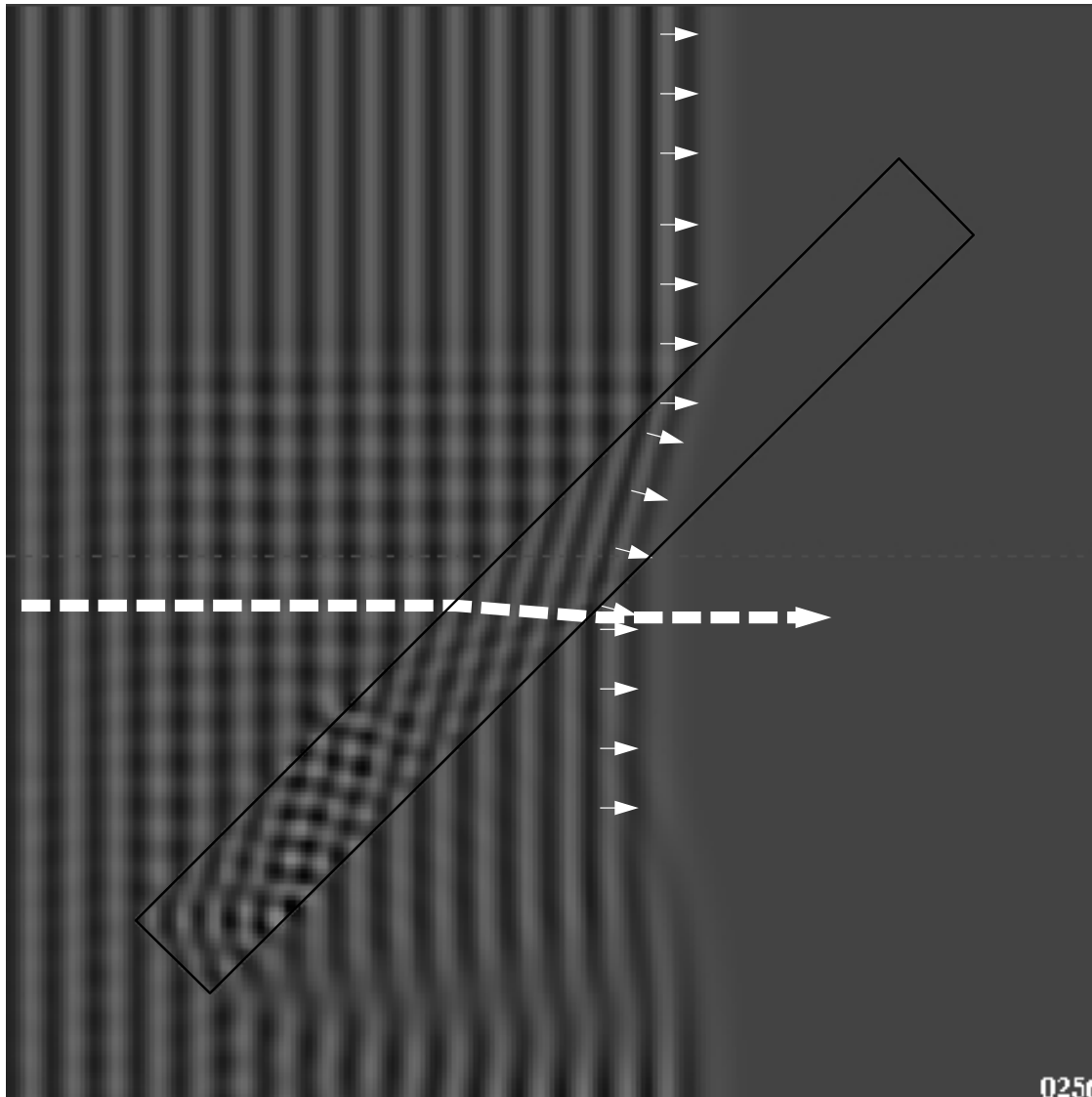
Diffraction

Diffraction is a wave phenomenon in which waves passing through small holes, around sharp corners spread out. Words such as "small" and "sharp" are relative to the wavelength of the light wave. Scales very much larger have little effect. The barn door casts a sharp shadow, and the light passing through the open barn doorway is unaffected. Scales very much smaller than the wavelength also have little effect. A mirror ground to with an eighth of a wavelength of light is generally considered high optical quality. Radio waves, whose wavelengths are much larger than light, are affected by the open iron superstructure of bridges.

The effect of diffraction can be easily shown by creating a reflecting wall with a small opening as shown in figure 7.16. As the wave fronts reach the wall, only a tiny portion may pass through the opening. Once through, the wave front then begins to spread, or diverge, as if the opening were itself a point source
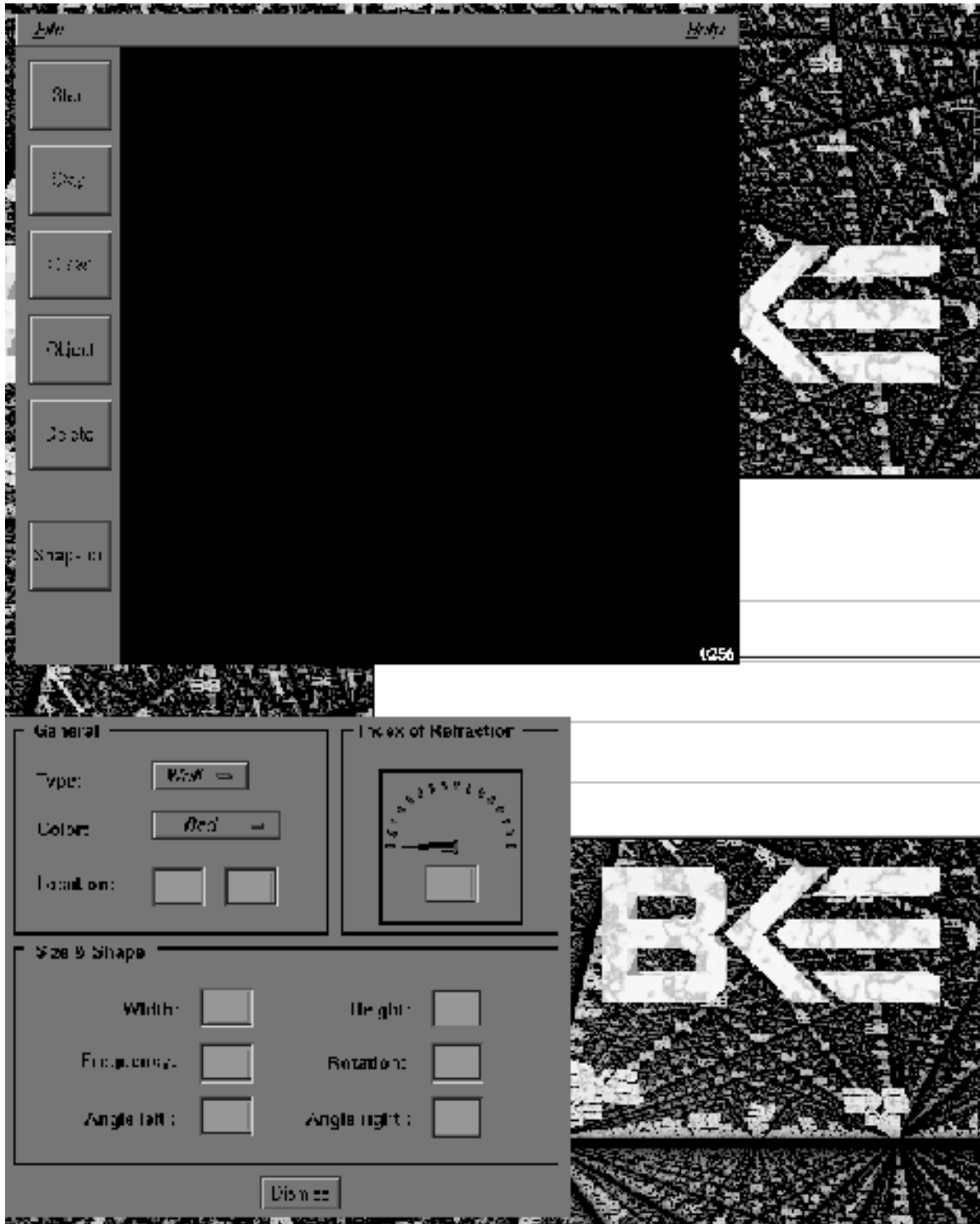
Figure 7.9 *These are the three windows created by the Wave Table program. The viewing window, in which a 2D image of the waves will be displayed is in the top−left. The control panel is in the bottom−left, and in the middle−right is a graphical window in which the amplitudes along a horizontal slice will be displayed.*

of light, figure 7.17. This is an illustration of *Huygens's Principle* that each portion of a wave front acts as a source of waves.

Figure 7.11 *This is an image from the wave table simulation with an intial configuration consisting of a high index of refraction slab that is positioned at 45° near the center of the table. Sinusoidal plane waves enter from the left edge of the table travelling to the right.When waves reach the surface of the slab, they are split into reflected waves (travelling upwards) and transmitted waves. As each wavefront enters the slab in which the speed of light is slower than in the external medium (here assumed vacuum), it must itself slow. Thus that portion of the wavefront falls behind the the wavefront in vacuum. The waves in the slab must continue at this slower speed until reaching the far edge of the slab, where waves are again split into reflected and transmitted waves. The transmitted waves resume the previous wave speed, but now trailing the positions they would otherwise have had. It should be noted that <u>wave fronts</u> are shown rather than the <u>rays</u> commonly encountered in textbooks. The rays are constructed from joining the instantaneous direction vectors along the wave fronts These instantaneous direction vectors are depicted as short white arrows for one of the wavefronts. A ray is constructed by joining adjacent vectors into one continuous ray, an example of which is shown by the white dashed line.*

Young's double slit experiment can be performed using the configuration shown in figure 7.18. Now each opening will act as a separate light source, but in phase with one another because they are initiated by
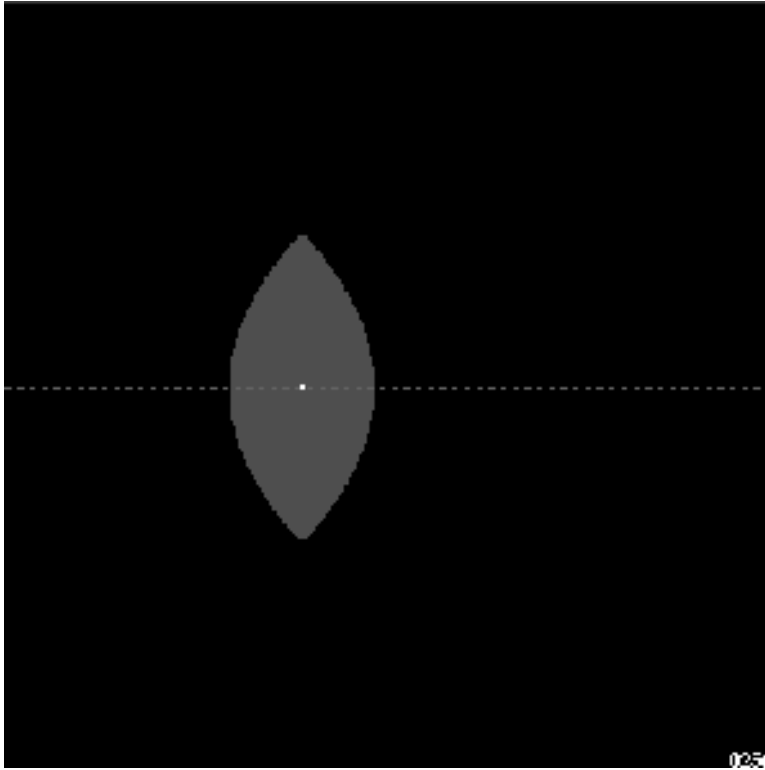
Figure 7.12 *This is the 2D viewing window showing the initial configuration for a wave table simulation demonstrating the properties of a converging lens. A high index of refraction converging lens is situated at the center of the table. Sinusoidal plane waves will enter from the left edge of the table travelling to the right.*

the same wave front. The resulting wave pattern is the result of the alternating interference between the two wave sources. It is analogous to the pattern observed when two pebbles are plunked into water, figure 7.19.
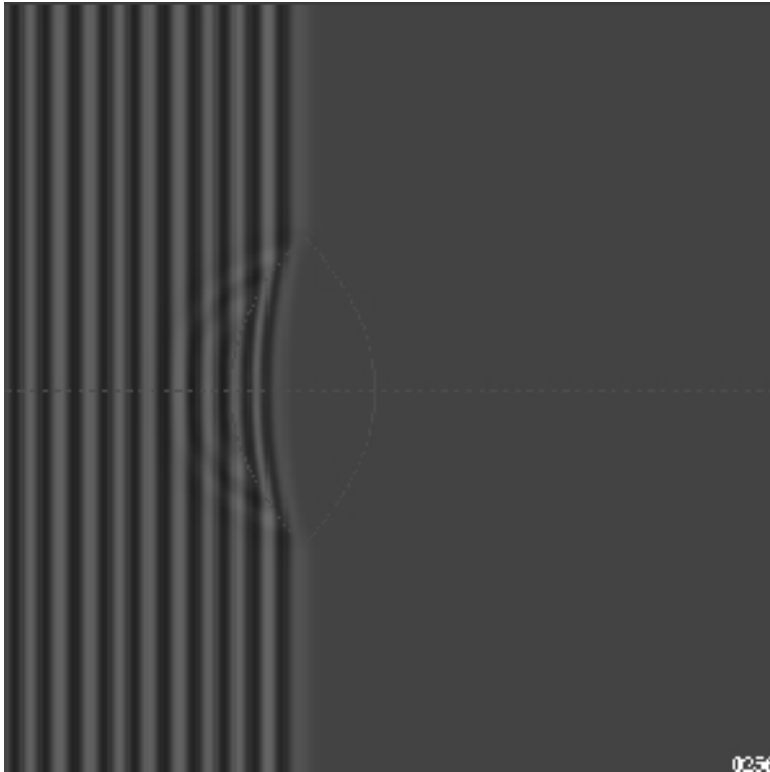
Figure 7.13 *These figures show the progress of a train of wavefronts through a converging lens. A (top): The center of the wave front enters the lens first and slows. Additional portions of the wave front enters the lens at progressively later times. B (bottom): he resulting curved wave front converges to a point.*
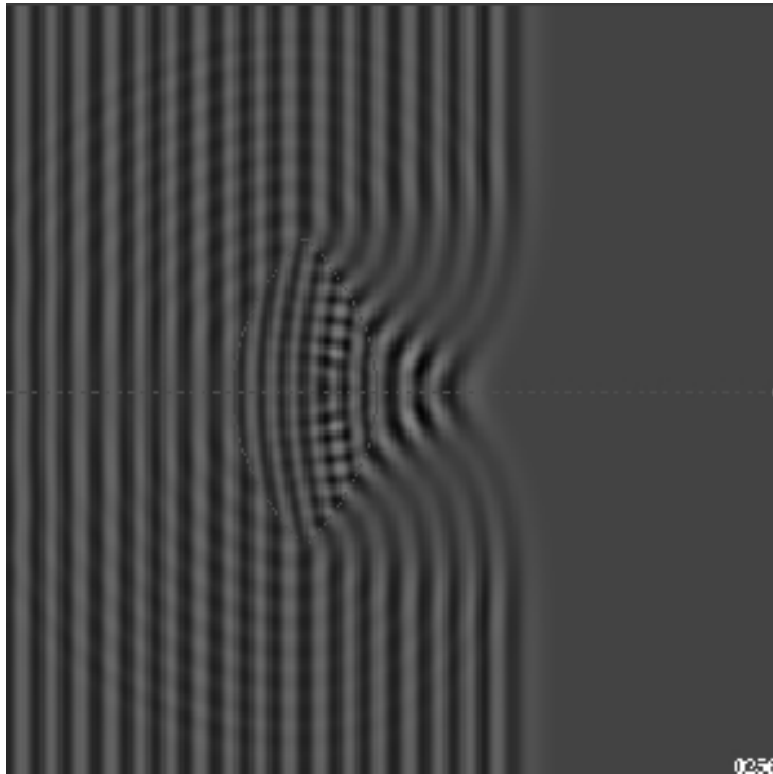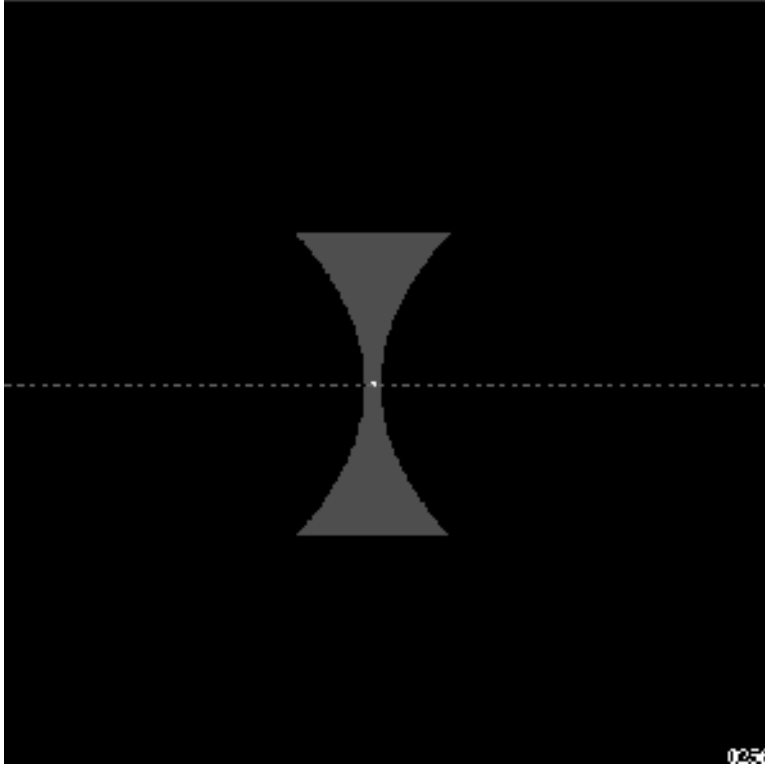
Figure 7.14 This is the 2D viewing window showing the initial configuration for a wave table simulation demonstrating the properties of a diverging lens. A high index of refraction diverging lens is situated at the center of the table. Sinusoidal plane waves will enter from the left edge of the table travelling to the right.
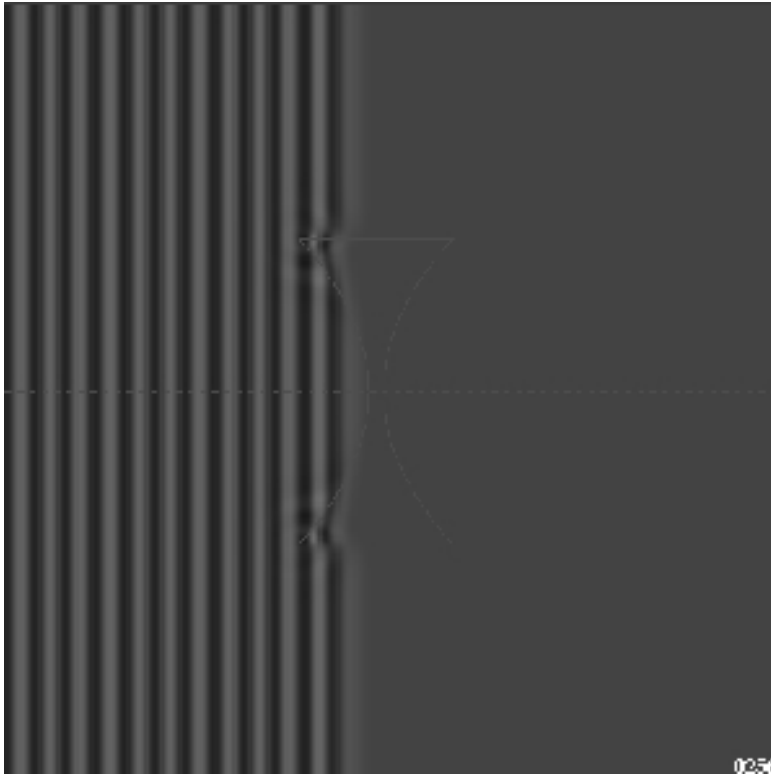
Figure 7.15 These figures show the progress of a train of wavefronts through a diverging lens. A (top): The waves enter the top and bottom of the lens before entering the lens center. The wave speed is slower in the lens, therefore those portions of the wave front entering first must slow first. Additional portions of the wave front enter the lens at progressively later times. B (bottom): This results in a curved, diverging wave front .
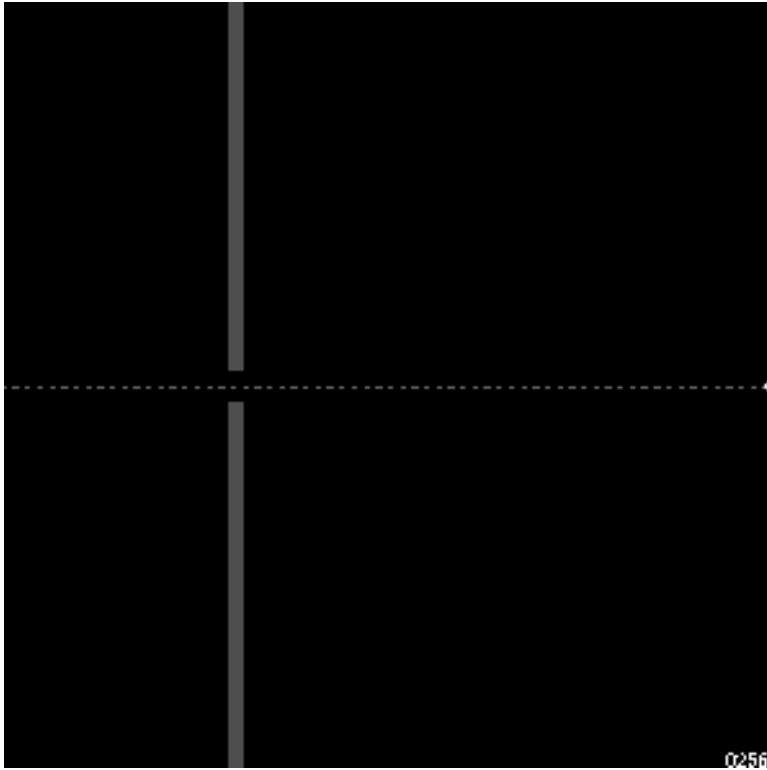
Figure 7.16   *This is the 2D viewing window showing the initial configuration for a wave table simulation for demonstrating single slit diffraction. Two opaque walls are placed in the left portion of the wave table with a small space between them which will form the slit. Sinusoidal plane waves will enter from the left edge of the table travelling to the right.*

Figure 7.17 *As wave fronts reach the wall, only a tiny portion of each may pass through the opening. Once, through, the wave fronts then begin to spread, or diverge, as if the opening was itself a point source of light. This is an illustration of <u>Huygens's Principle</u> that each portion of a wave front acts as a source of waves. In the reqion between the left edge of the wave table and each wall exist <u>standing waves</u> where the leftward moving reflected plane waves interfere with waves travelling to the right producing a spatially stable pattern.*
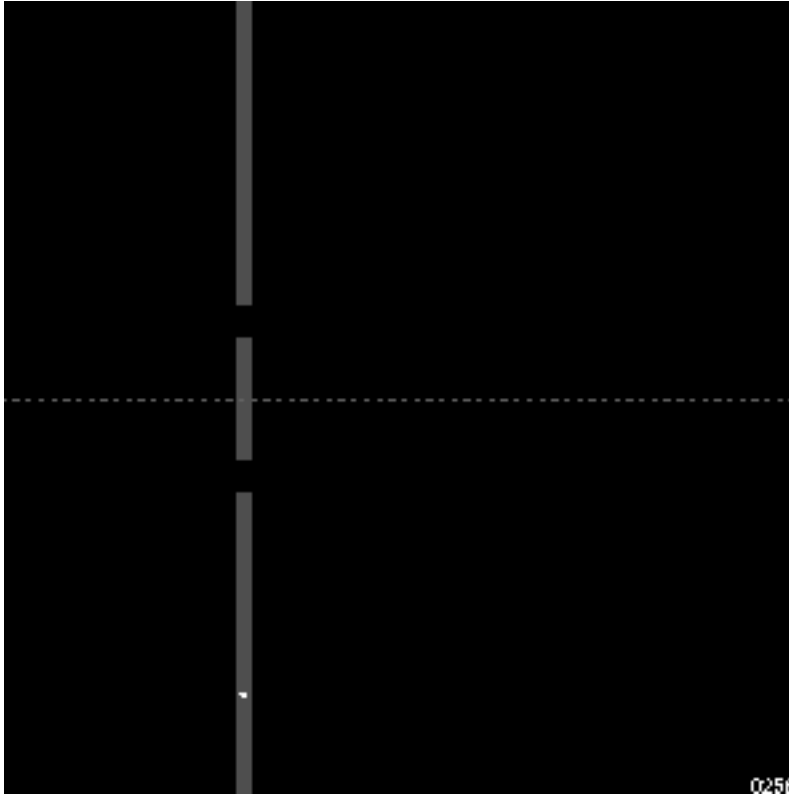
Figure 7.18 *This is the 2D viewing window showing the initial configuration for a wave table simulation for demonstrating Young's double slit experiment. Three opaque walls are used to create two slits in the left portion of the wave table. Sinusoidal plane waves will enter from the left edge of the table travelling to the right.*
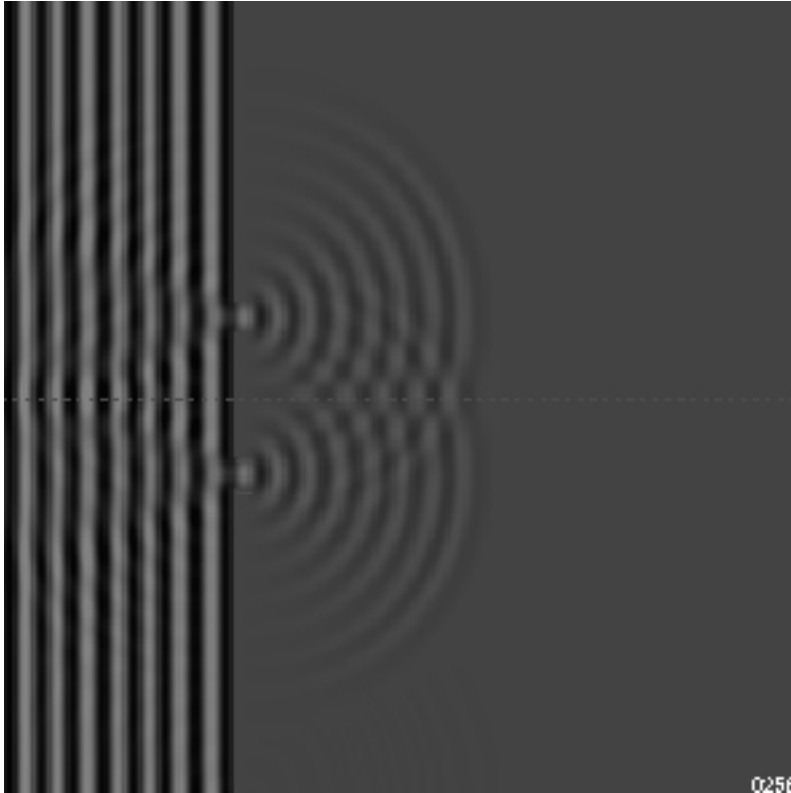
Figure 7.19   In Young's double slit experiment, two openings exist, each of which will act as a separate light source, but in phase with one another, because they are initiated by the same wave front. The resulting wave pattern is the result of the alternating interference between the two wave sources. It is analogous to the pattern observed when two pebbles are plunked into water.

Tables

Table 1			
Iteration	time	y^n	v^n
0	0.00000	1.00000	0.00000
1	0.50000	0.87500	-0.50000
2	1.00000	0.51563	-0.93750
3	1.50000	-0.01758	-1.19531
4	2.00000	-0.61304	-1.18652
5	2.50000	-1.12967	-0.88000
6	3.00000	-1.42846	-0.31517
7	3.50000	-1.40749	0.39906
8	4.00000	-1.03202	1.10281
9	4.50000	-0.35162	1.61882
10	5.00000	0.50174	1.79463
11	5.50000	1.33634	1.54375
12	6.00000	1.94117	0.87559
13	6.50000	2.13632	-0.09500
14	7.00000	1.82178	-1.16316
15	7.50000	1.01248	-2.07405
16	8.00000	-0.15111	-2.58029
17	8.50000	-1.42237	-2.50473
18	9.00000	-2.49694	-1.79355
19	9.50000	-3.08160	-0.54508
20	10.00000	-2.96894	0.99571
21	10.50000	-2.09996	2.48018
22	11.00000	-0.59738	3.53017
23	11.50000	1.24238	3.82885
24	12.00000	3.00151	3.20766
25	12.50000	4.23015	1.70691

TABLE 1: This shows results of using the unstable procedure given by eqns 2.11:

$$v^{n+1} = v^n - \frac{k}{m}(x^n - x_0)\Delta t \qquad 2.11a$$

and

$$x^{n+1} = x^n + \frac{(v^n + v^{n+1})}{2}\Delta t \qquad 2.11b$$

described in the text. The initial conditions are: $x_0 = 0$, $x^0 = 1$, and $v^0 = 0$. The timestep is $\Delta t = 0.5$. Note that the amplitude increases with each oscillation, betraying the instability (figure 2a).

Table 2			
Iteration	time	y^n	v^n
0	0.00000	1.00000	0.00000
1	0.25000	0.96875	-0.25000
2	0.50000	0.87598	-0.49219
3	0.75000	0.72556	-0.71118
4	1.00000	0.52509	-0.89257
5	1.25000	0.28553	-1.02384
6	1.50000	0.02065	-1.09523
7	1.75000	-0.25380	-1.10039
8	2.00000	-0.52097	-1.03694
9	2.25000	-0.76392	-0.90670
10	2.50000	-0.96672	-0.71572
11	2.75000	-1.11544	-0.47404
12	3.00000	-1.19909	-0.19518
13	3.25000	-1.21042	0.10460
14	3.50000	-1.14644	0.40720
15	3.75000	-1.00881	0.69381
16	4.00000	-0.80384	0.94601
17	4.25000	-0.54221	1.14697
18	4.50000	-0.23852	1.28253
19	4.75000	0.08956	1.34216
20	5.00000	0.42230	1.31977
21	5.25000	0.73905	1.21419
22	5.50000	1.01950	1.02943
23	5.75000	1.24500	0.77456
24	6.00000	1.39973	0.46331
25	6.25000	1.47182	0.11337
26	6.50000	1.45416	-0.25458
27	6.75000	1.34508	-0.61812
28	7.00000	1.14851	-0.95439
29	7.25000	0.87402	-1.24152
30	7.50000	0.53633	-1.46003
31	7.75000	0.15456	-1.59411
32	8.00000	-0.24879	-1.63275
33	8.25000	-0.64921	-1.57055
34	8.50000	-1.02156	-1.40825
35	8.75000	-1.34169	-1.15286
36	9.00000	-1.58798	-0.81744
37	9.25000	-1.74272	-0.42044
38	9.50000	-1.79337	0.01524
39	9.75000	-1.73351	0.46358
40	10.00000	-1.56345	0.89696
41	10.25000	-1.29035	1.28782
42	10.50000	-0.92807	1.61041
43	10.75000	-0.49647	1.84242
44	11.00000	-0.02035	1.96654
45	11.25000	0.47192	1.97163
46	11.50000	0.95008	1.85365
47	11.75000	1.38381	1.61613
48	12.00000	1.74459	1.27018
49	12.25000	2.00762	0.83403
50	12.50000	2.15339	0.33212

TABLE 2: This shows results of using the unstable procedure given by eqns 2.11:

$$v^{n+1} = v^n - \frac{k}{m}(x^n - x_0)\Delta t \qquad\qquad\qquad 2.11a$$

and

$$x^{n+1} = x^n + \frac{(v^n + v^{n+1})}{2}\Delta t \qquad\qquad\qquad 2.11b$$

described in the text. This is the same procedure used for the results shown in table 1, as are the initial conditions: $x_0 = 0$, $x^0 = 1$, and $v^0 = 0$. However, the timestep is $\Delta t = 0.25$, half that used for the results in Table 1. Note that the amplitude still increases with each oscillation, although at a slower rate (figure 2). This shows that using eqns 2.11 produces an unstable result, regardless of how small the timestep is made.

Table 3			
Iteration	time	y^n	v^n
0	0.00000	1.00000	0.00000
1	0.25000	0.96875	-0.25000
2	0.50000	0.87598	-0.49219
3	0.75000	0.72556	-0.71118
4	1.00000	0.52509	-0.89257
5	1.25000	0.28553	-1.02384
6	1.50000	0.02065	-1.09523
7	1.75000	-0.25380	-1.10039
8	2.00000	-0.52097	-1.03694
9	2.25000	-0.76392	-0.90670
10	2.50000	-0.96672	-0.71572
11	2.75000	-1.11544	-0.47404
12	3.00000	-1.19909	-0.19518
13	3.25000	-1.21042	0.10460
14	3.50000	-1.14644	0.40720
15	3.75000	-1.00881	0.69381
16	4.00000	-0.80384	0.94601
17	4.25000	-0.54221	1.14697
18	4.50000	-0.23852	1.28253
19	4.75000	0.08956	1.34216
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29	7.25000	0.87402	-1.24152
30	7.50000	0.53633	-1.46003
31	7.75000	0.15456	-1.59411
32	8.00000	-0.24879	-1.63275
33	8.25000	-0.64921	-1.57055
34	8.50000	-1.02156	-1.40825
35	8.75000	-1.34169	-1.15286
36	9.00000	-1.58798	-0.81744
37	9.25000	-1.74272	-0.42044
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45	11.25000	0.47192	1.97163
46	11.50000	0.95008	1.85365
47	11.75000	1.38381	1.61613
48	12.00000	1.74459	1.27018
49	12.25000	2.00762	0.83403
50	12.50000	2.15339	0.33212

TABLE 3: This shows results of using the stable leapfrog procedure given by eqns 2.12:

$$v^{n+\frac{1}{2}} = v^{n-\frac{1}{2}} + \frac{k}{m}(x^n - x_0)\,\Delta t \qquad\qquad (2.12a)$$

and

$$x^{n+1} = x^n + v^{n+\frac{1}{2}}\,\Delta t, \qquad\qquad (2.12b)$$

as described in the text. The initial conditions are: $x^0 = 1$ and $v^{-\frac{1}{2}} = 0$. The timestep is $\Delta t = 0.5$. Note that the error introduced by using $v^{-\frac{1}{2}} = 0$ for the velocity $tv^{-\frac{1}{2}}$ causes the magnitude of the amplitude to be about 1.03 rather than the expected 1. (figure 2) This starting error does not grow.