7. Two Dimensions

a Theory I - Isolated mass between four springs

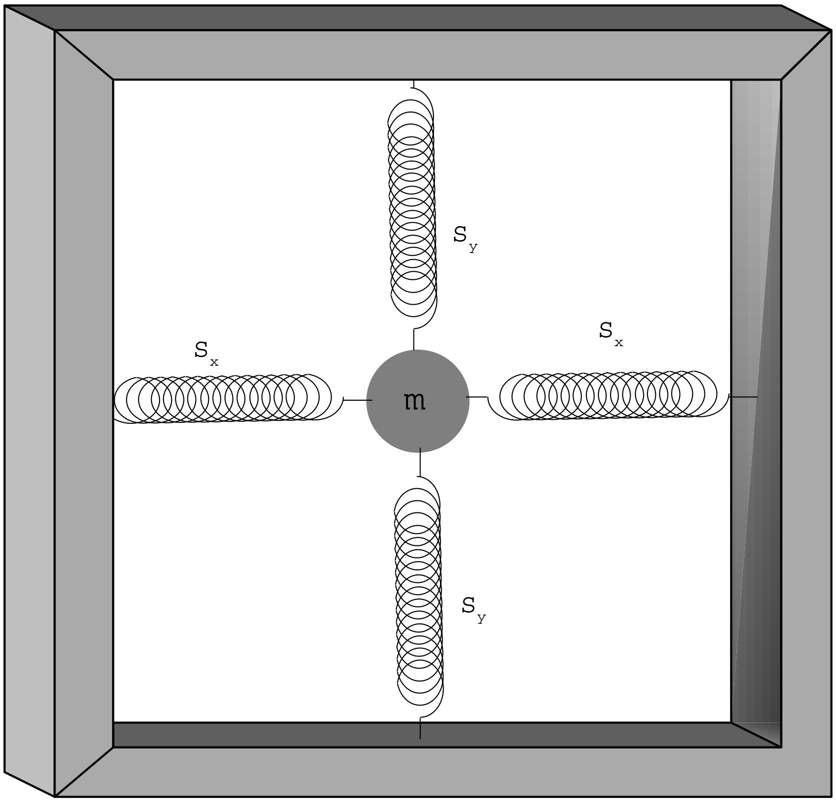

Figure 7.1 *This shows a mass suspended by two pairs of identical springs within a rectangular frame. One spring pair lies along the x−axis and the other along the y−axis. They exert sufficient force on the central mass so that gravity may be neglected.*

The one-dimensional discussion above is easily extended to two dimensions. Consider the configuration shown in figure 7.1. Two pairs of identical springs suspend a single mass within a rectangular frame. One spring pair lies along the x-axis and the other along the y-axis. They exert sufficient force on the central mass so that gravity may be neglected. Now displace the mass along the z-axis and examine the resulting force on the mass (figure 7.2). As with the one-dimensional example, the x-components of the force cancel, as do the y-components, leaving only a vertical component:

$$F_z = -2F_x \sin\theta - 2F_y \sin\phi \tag{7.1}$$

where z is the distance from equilibrium at $z = 0$, θ is the angle the springs parallel to the x-axis makes with the horizontal, and ϕ is the angle the springs parallel to the y-axis makes with the horizontal. It is assumed that the displacement is small compared to the lengths of the strings, that is,

$$\begin{gathered} |z| \ll \Delta x \\ \text{and} \\ |z| \ll \Delta y. \end{gathered} \tag{7.2}$$

Then eqn (7.1) becomes

$$F_z \approx -2(F_x + F_y)z. \tag{7.3}$$

Thus the mass undergoes simple harmonic motion along the z-axis as if attached to a single spring with an effective spring constant of

$$k' = 2(F_x + F_y) \tag{7.4}$$

b Theory II - An Array of Masses

Now consider a two-dimensional array of masses and springs, as in figure 7.3. All springs parallel to the x-axis are identical to one another, and all springs parallel to the y-axis are identical to one another. It

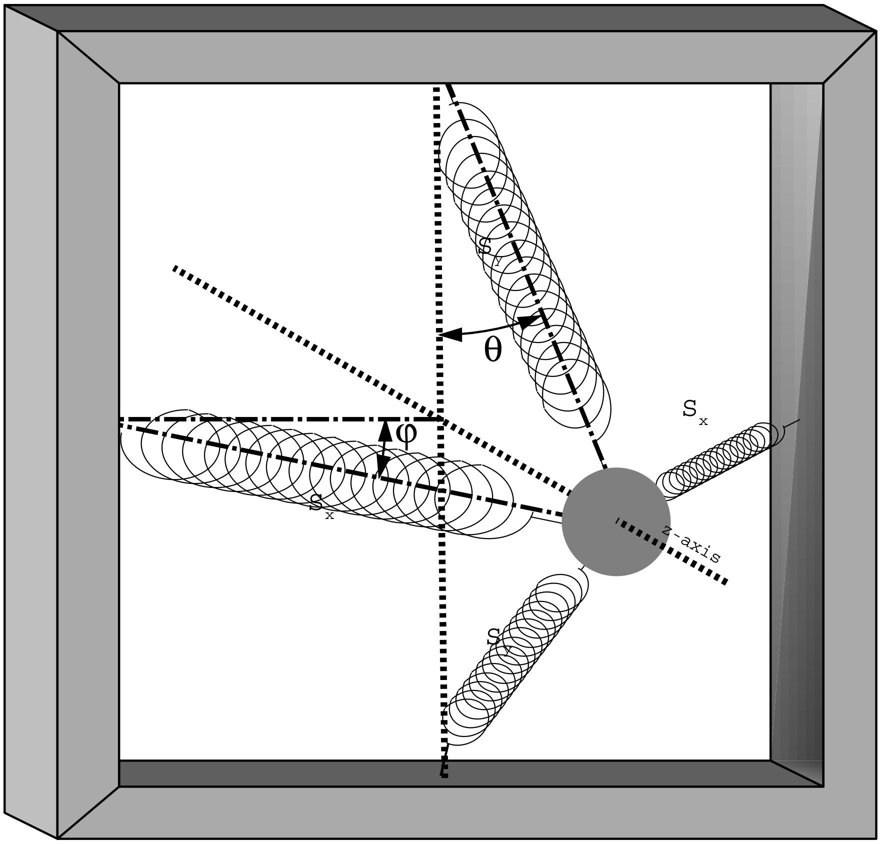

Figure 7.2 *The mass suspended by two pairs of identical springs within a rectangular frame from figure 7.1 is displaced a short distance along the z−axis from the equilibrium position.*

is assumed that the spacings, Δx and Δy, are equal and any vertical displacements will be small relative to this spacing.

In order to study the forces on the *(i,j)*th mass, first examine figure 7.4a which shows the *j*th row from fig. 7.3 with a small *z* displacement. The horizontal force components will cancel for small displacements, leaving only a vertical component

$$f_{i,j} = F_x \Big[\sin(\theta_l) - \sin(\theta_r) \Big], \tag{7.5}$$

where F_x is the magnitude of the force the each of the two springs parallel to the x-axis exerts on the individual masses, θ_l is the angle the spring to the left of the mass makes with respect to the horizontal, and θ_r is the angle the spring to the right of the mass makes with respect to the horizontal. From the figure it is clear that

$$\sin(\theta_l) = \frac{z_{i,j} - z_{i-1,j}}{\sqrt{(\Delta x)^2 + (z_{i,j} - z_{i-1,j})^2}} \approx \frac{z_{i,j} - z_{i-1,j}}{\Delta x} \tag{7.6}$$

and

$$\sin(\theta_r) = \frac{z_{i+1,j} - z_{i,j}}{\sqrt{\Delta x^2 + (z_{i+1,j} - z_{i,j})^2}} \approx \frac{z_{i+1,j} - z_{i,j}}{\Delta x}\,. \tag{7.8}$$

Thus the contribution to the force acting on $m_{i,j}$ due to the springs parallel to the x-axis is

$$f_{i,j} = F_x \left(\frac{z_{i+1,j} - z_{i,j}}{\Delta x} - \frac{z_{i,j} - z_{i-1,j}}{\Delta x} \right). \tag{7.9}$$

Now inspect figure 7.4b which shows the *i*th column from fig. 7.3 with a small *z* displacement. Again, the horizontal force components cancel, leaving only a vertical component

$$g_{i,j} = F_y \Big[\sin(\phi_l) - \sin(\phi_r) \Big], \tag{7.10}$$

where F_y is the magnitude of the force the each of the two springs parallel to the y-axis exerts on the

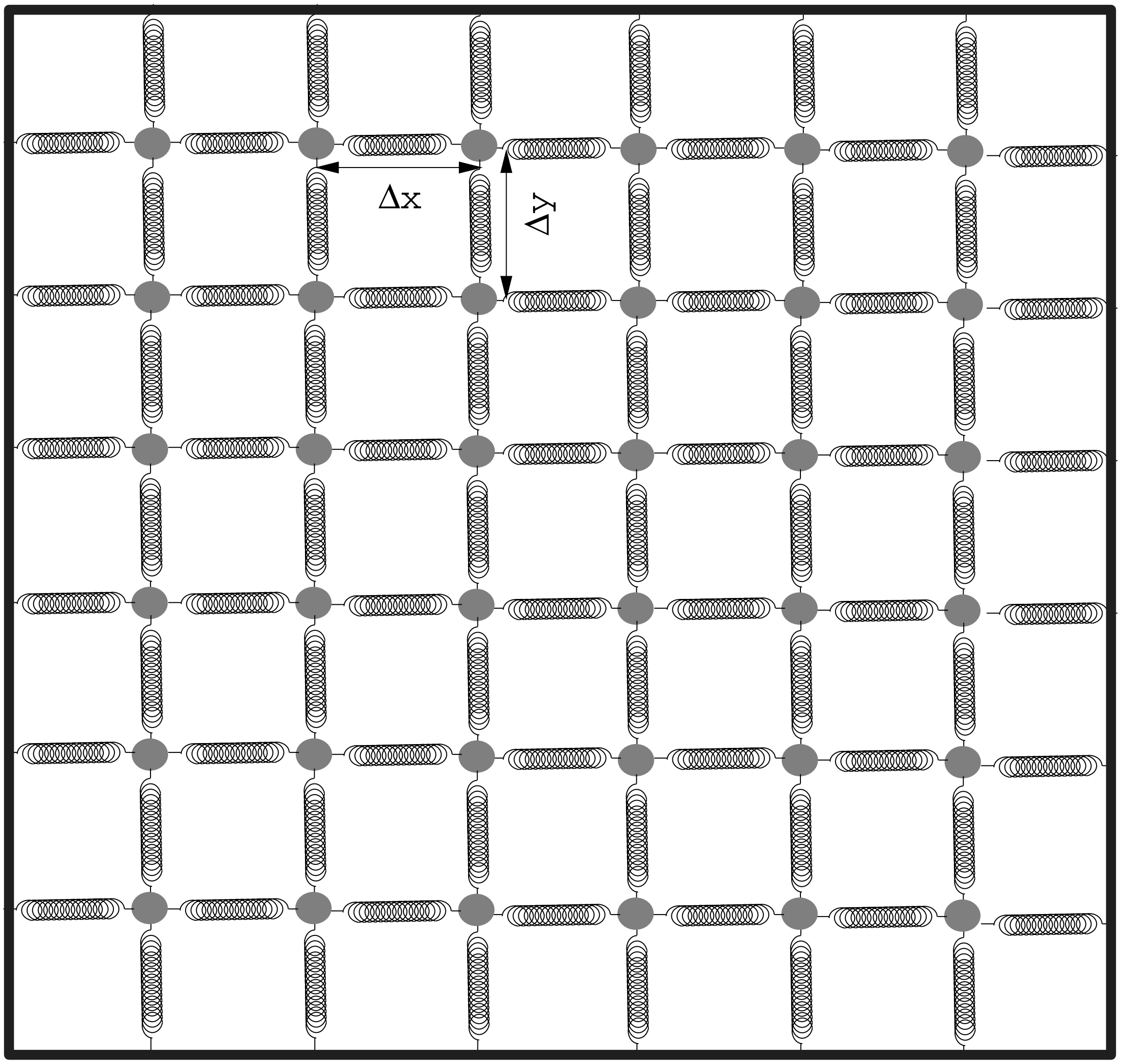

Figure 7.3 This shows a two dimensional array of masses and springs in a rectangular frame. The springs exert enough force so that no significant sagging exists at equilibrium.

individual masses, ϕ_l is the angle the spring to the left of the mass makes with respect to the horizontal, and ϕ_r is the angle the spring to the right of the mass makes with respect to the horizontal. It is clear that

$$\sin(\phi_l) = \frac{z_{i,j} - z_{i,j-1}}{\sqrt{(\Delta y)^2 + (z_{i,j} - z_{i,j-1})^2}} \approx \frac{z_{i,j} - z_{i,j-1}}{(\Delta y)} \tag{7.11}$$

and

$$\sin(\phi_r) = \frac{z_{i,j+1} - z_{i,j}}{\sqrt{(\Delta y)^2 + (z_{i,j+1} - z_{i,j})^2}} \approx \frac{z_{i,j+1} - z_{i,j}}{(\Delta y)}\ . \tag{7.12}$$

The contribution to the force acting on $m_{i,j}$ due to the springs parallel to the y-axis is then

$$g_{i,j} = F_y \left[\frac{z_{i,j+1} - z_{i,j}}{\Delta y} - \frac{z_{i,j} - z_{i,j-1}}{\Delta y} \right]. \tag{7.13}$$

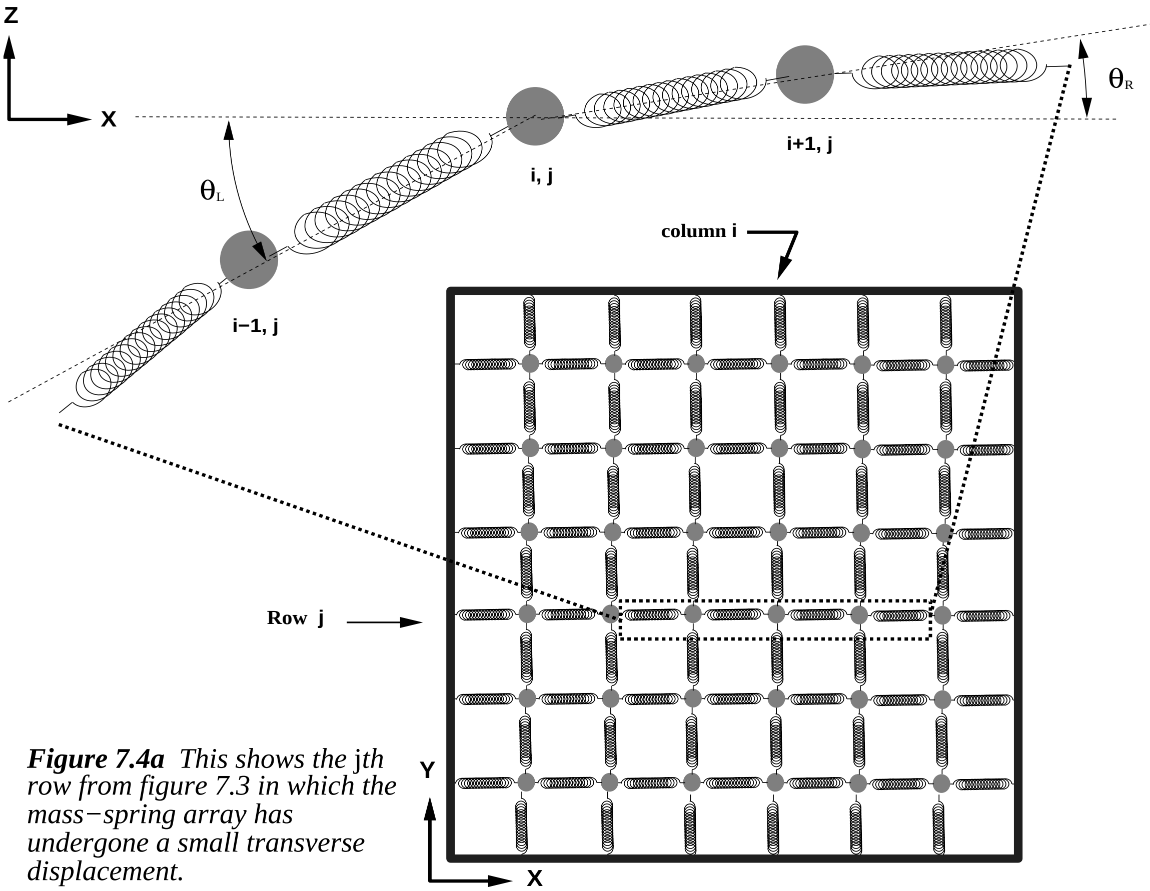

Figure 7.4a *This shows the* j*th row from figure 7.3 in which the mass−spring array has undergone a small transverse displacement.*

Thus the acceleration on the i,jth mass becomes

$$a_{i,j} = \frac{\left(f_{i,j} + g_{i,j}\right)}{m_{i,j}} = \frac{F_x}{m_{i,j}}\left[\frac{z_{i+1,j} - z_{i,j}}{\Delta x} - \frac{z_{i,j} - z_{i-1,j}}{\Delta x}\right] + \frac{F_y}{m_{i,j}}\left[\frac{z_{i,j+1} - z_{i,j}}{\Delta y} - \frac{z_{i,j} - z_{i,j-1}}{\Delta y}\right], \qquad (7.14)$$

The remainder of this discussion will make use of the simplifying assumptions that all springs are identical,

$$F_x = F_y = F, \qquad (7.15)$$

and that the spacings in x and y are equal,

$$\Delta y = \Delta x. \qquad (7.16)$$

Constructing an algorithm for the non-isotropic case is straightforward, and is left as an exercise for the interested reader. Using the above assumptions, eqn. (7.14) becomes

$$\frac{F_y}{m_{i,j}}\left[\frac{z_{i+1,j} + z_{i,j+1} - 4\,z_{i,j} + z_{i-1,j} + z_{i,j-1}}{\Delta x}\right], \qquad (7.17)$$

and a numerical solution can be constructed using eqns (2.12):

$$v_{i,j}^{n+1/2} = v_{i,j}^{n-1/2} + \frac{F}{m_{i,j}\,\Delta x}\,(z_{i-1,j}^{n} + z_{i,j-1}^{n} - 4\,z_{i,j}^{n} + (z_{i+1,j}^{n} + z_{i,j+1}^{n})\,\Delta t \qquad (7.18a)$$

and

$$z_{i,j}^{n+1} = z_{i,j}^{n} + v_{i,j}^{n+1/2}\,\Delta t. \qquad (7.18b)$$

Obviously, the initial conditions must specify the starting displacements and velocities for each mass. To complete *one* timestep, eqns (7.18) must be computed $N_x * N_y$ times. That is, once for each mass.

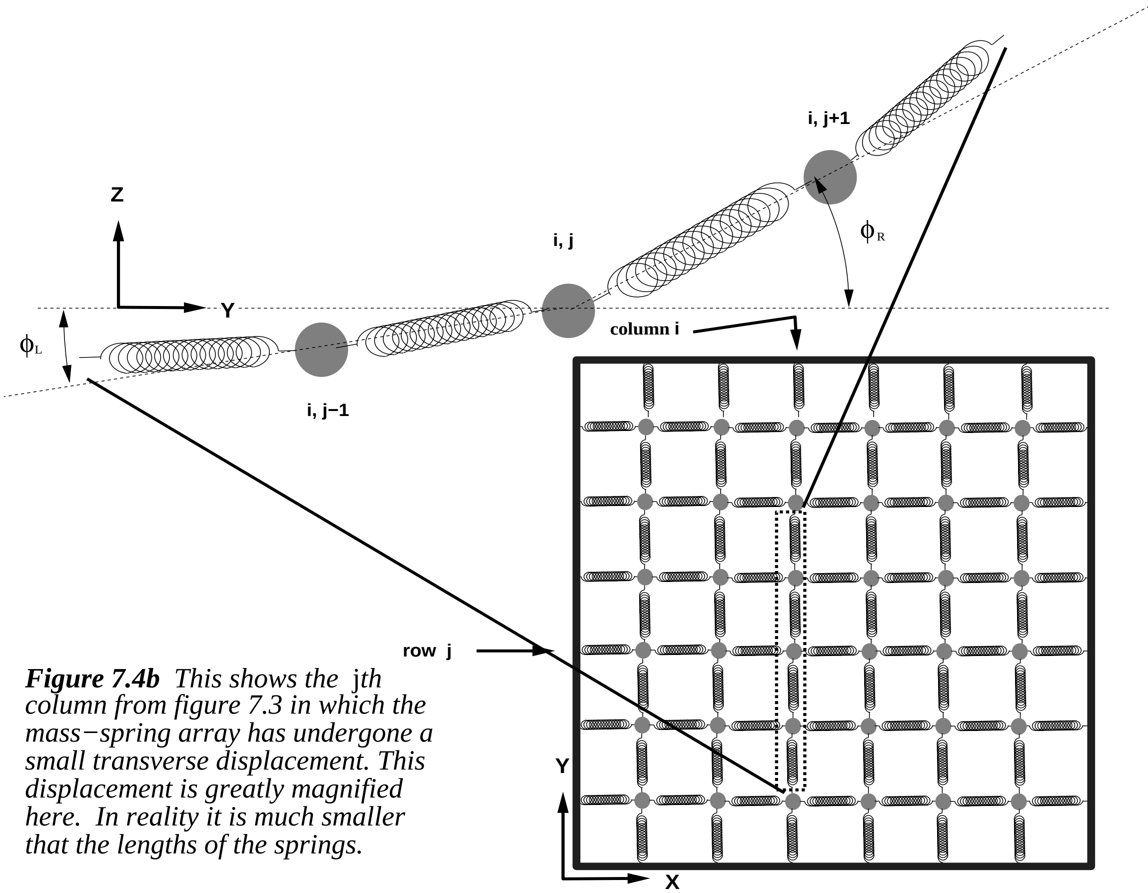

Figure 7.4b *This shows the jth column from figure 7.3 in which the mass−spring array has undergone a small transverse displacement. This displacement is greatly magnified here. In reality it is much smaller that the lengths of the springs.*

c Theory III - A Stretched Membrane

Now consider a membrane stretched on a rectangular frame. Here, a ‘‘membrane’’ will be considered to be any flexible length of material whose area is large enough so that its thickness is insignificant when compared to its length or width. Its only defining parameter is its surface density, σ. It will be assumed to be clamped at its ends and ‘‘stretched’’ by forces pulling on the each of its ends. It will be further assumed that these edge forces pull perpendicular to the edges, so no shear forces are introduced. This creates tension in the membrane.

In a membrane, tension has units of force per unit length, as opposed to string tension which has units of force. To understand the source of the difference, consider figure 7.5 which depicts one method for creating a specific tension in a string. A string is attached to a rigid wall. A weight of mass m is attached to the other end, with the string passing over a pulley. The mass of the string is much less than the weight. The tension in the string is equal to the downward force produced by the weight,

$$F = m\,g, \tag{7.19}$$

where g is the gravitational acceleration. In figure 7.6a, the string has been replaced by two identical strings. The total force pulling on the strings remains the same, F, but this is now distributed over two strings, so the tension in each is half of that force or $F/2$. If, as in figure 7.6b, there are four strings, the tension per string is only $F/4$. This can be extended to N strings, for which the tension at any point in any string would be

$$T = \frac{F}{N}\,. \tag{7.20}$$

Now if a membrane of width W, figure 7.6c, was considered to be constructed of N strings (as if a woven fabric), the number of strings per unit length would be N/W, and the tension at a point in the membrane could be expressed as

$$T = \frac{F}{W}\,. \tag{7.21}$$

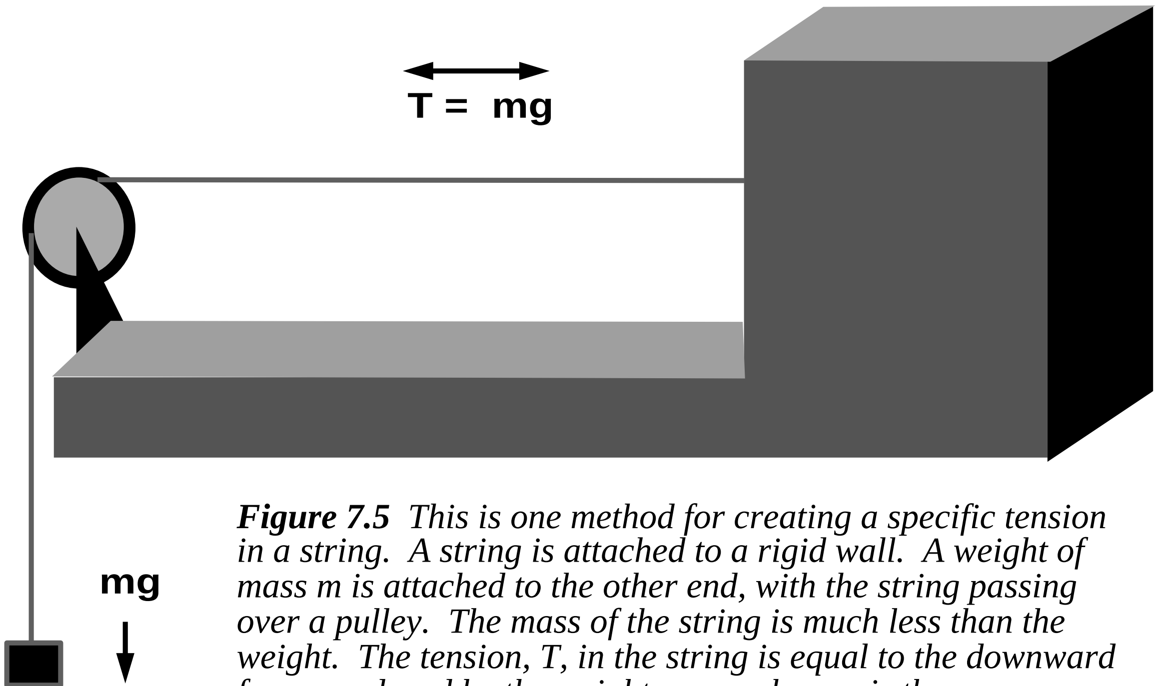

Figure 7.5 *This is one method for creating a specific tension in a string. A string is attached to a rigid wall. A weight of mass m is attached to the other end, with the string passing over a pulley. The mass of the string is much less than the weight. The tension, T, in the string is equal to the downward force produced by the weight, mg, where g is the gravitational acceleration.*

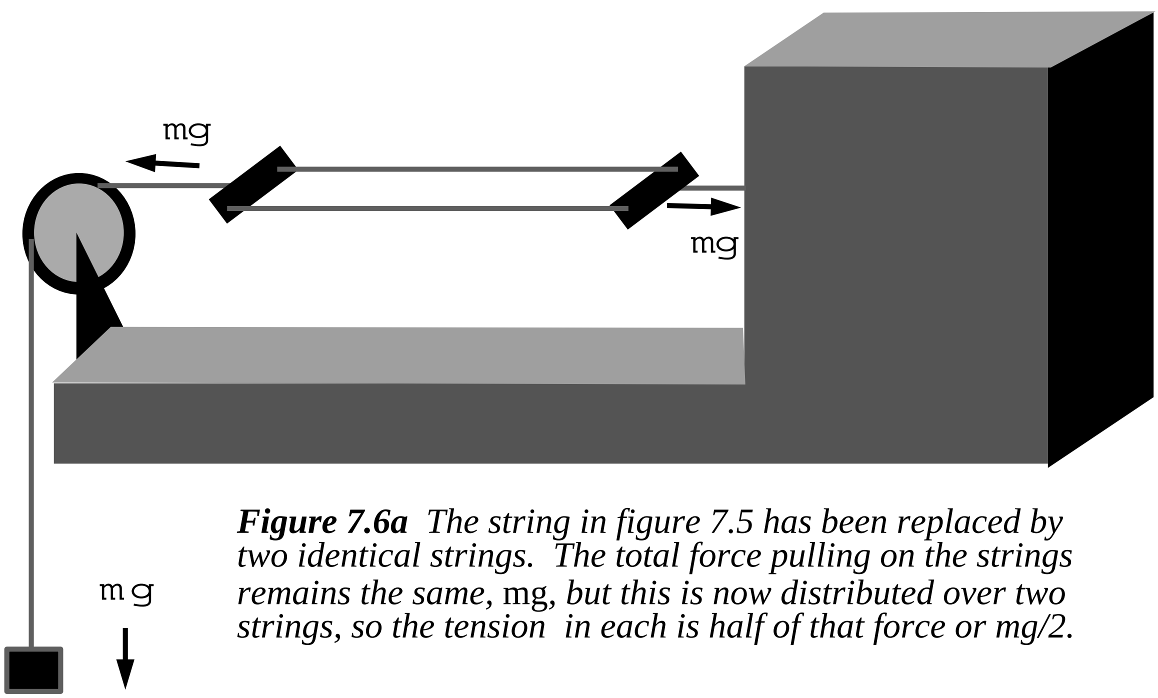

Figure 7.6a *The string in figure 7.5 has been replaced by two identical strings. The total force pulling on the strings remains the same,* mg, *but this is now distributed over two strings, so the tension in each is half of that force or mg/2.*

In general, the tensions in the two different directions may differ, but it will be assumed here that they are the same. Extending the following discussion for the case of differing tensions is straightforward, if tedious. It is assumed that enough tension is applied to prevent any significant sagging, so that gravity may be neglected.

Figure 7.7 depicts a rectangular stretched membrane with dimensions L_x and L_Y, divided into many small surface elements, each with area $\Delta x \, \Delta y$. These are assumed small compared to the overall

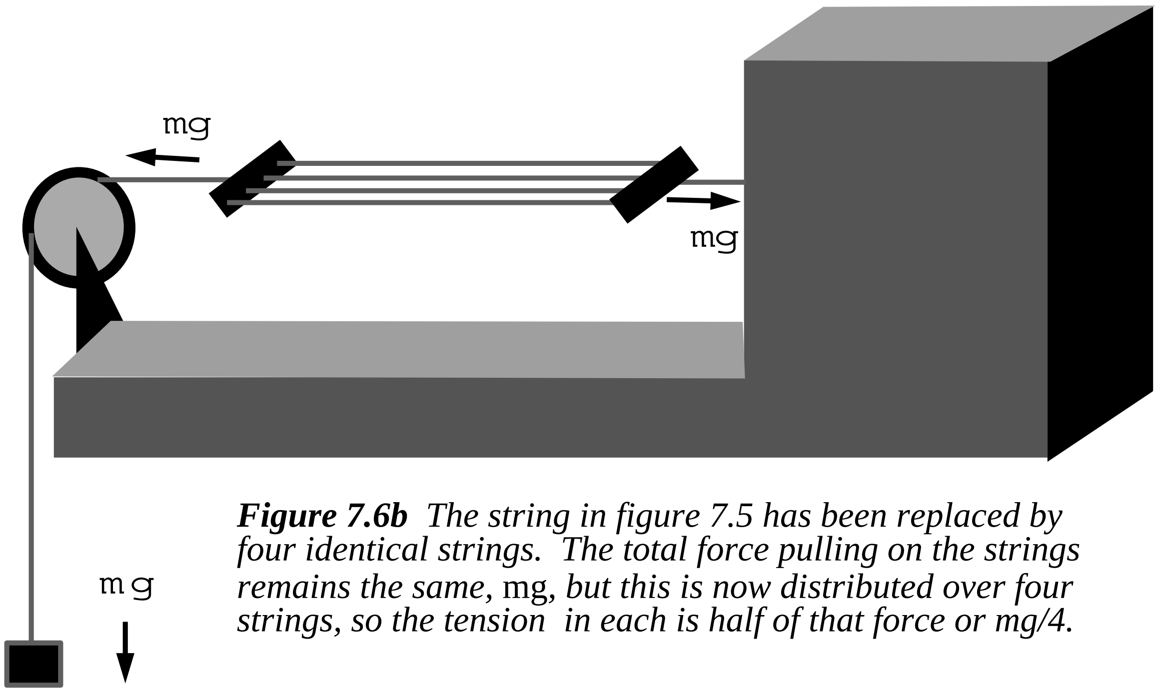

__Figure 7.6b__ The string in figure 7.5 has been replaced by four identical strings. The total force pulling on the strings remains the same, mg, *but this is now distributed over four strings, so the tension in each is half of that force or mg/4.*

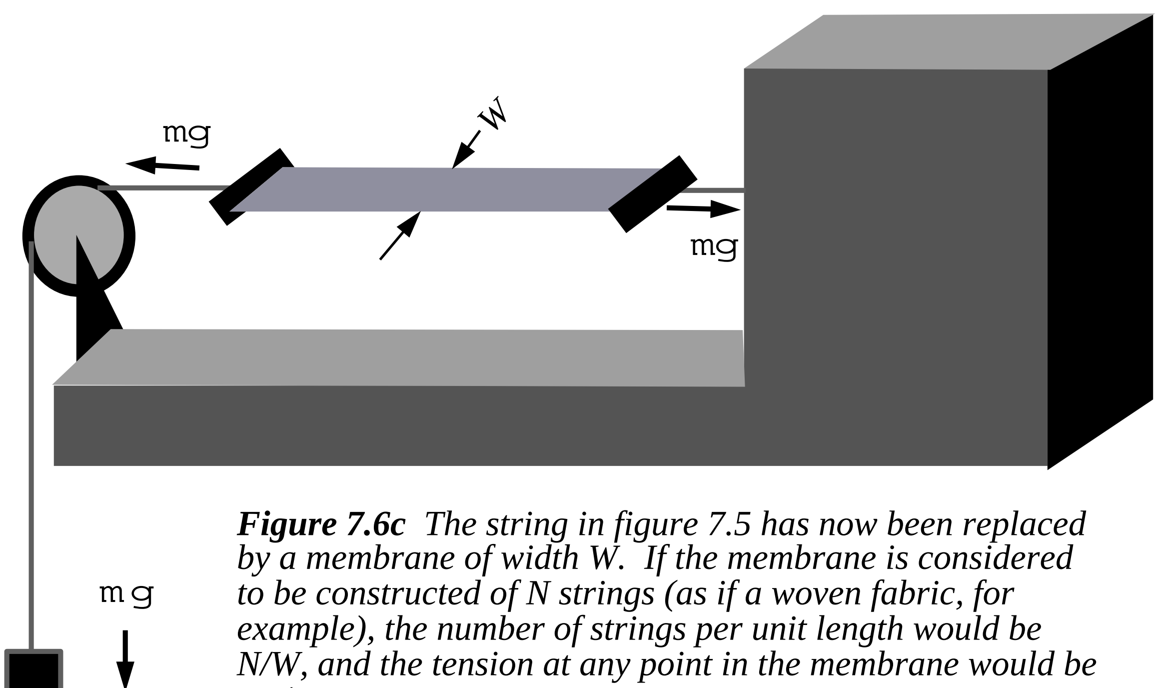

__Figure 7.6c__ The string in figure 7.5 has now been replaced by a membrane of width W. If the membrane is considered to be constructed of N strings (as if a woven fabric, for example), the number of strings per unit length would be N/W, and the tension at any point in the membrane would be mg/w.

dimensions of the membrane. That is

$$\Delta x \ll L_x \quad \text{and} \quad \Delta y \ll L_y. \tag{7.22}$$

Figure 7.8 shows one of the surface elements with four of its neighbors from a stretched membrane which has undergone a small displacement in the z-direction. Also shown are the four forces pulling on this element. (It is assumed that any tension along the diagonal is negligible.) When the displacements are small,

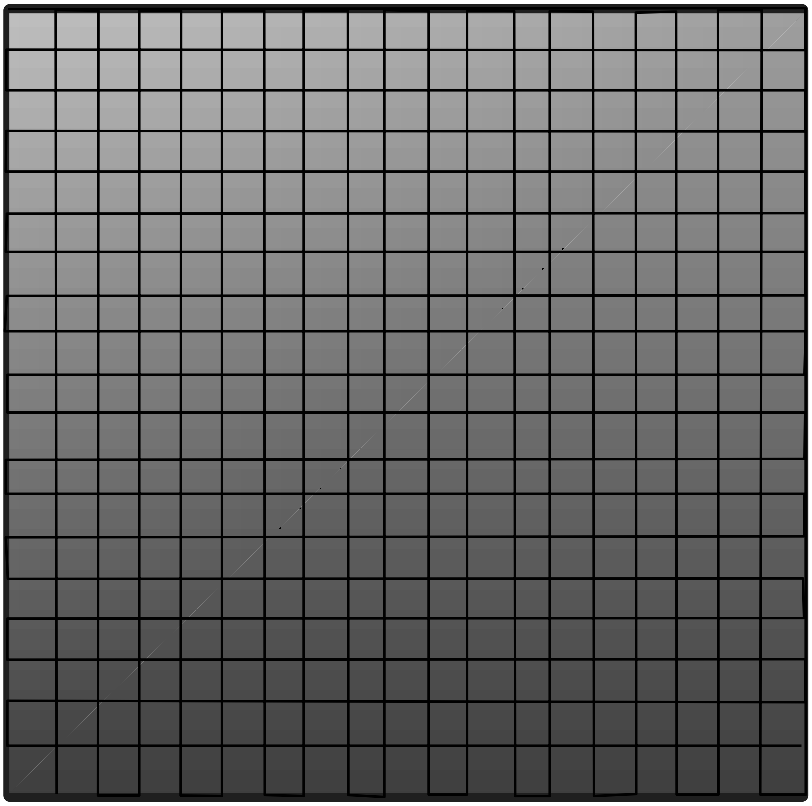

Figure 7.7 *This depicts a rectangular stretched membrane with dimensions L_x and L_y, divided into many small surface elements, each with area $\Delta x \cdot \Delta y$. These elements are assumed small compared to the overall dimensions of the membrane.*

all horizontal forces cancel, leaving only a vertical component. Furthermore, if the displacement is smooth, that is, it is not discontinuous, the directions of the four forces acting on the membrane may be approximated by the directions of the lines joining the center of element *(i,j)* and the centers of the four adjoining elements. The net contribution of the force pulling to the left is

$$F_L \sin \xi_L = T\, \Delta y \sin \xi_L. \tag{7.23}$$

The factor of Δy is required because the tension acts along a distance Δy (the left edge of the zone). Here, ξ_L is the angle between the line joining the centers of element (i,j) and element (i-1,j) with the horizontal, or

$$\sin \xi_L = \frac{z_{i-1,j} - z_{i,j}}{\sqrt{\Delta x^2 + (z_{i-1,j} - z_{i,j})^2}}\,. \tag{7.24}$$

The assumption of small vertical displacements implies

$$|z_{i-1,j} - z_{i,j}| \ll \Delta x, \tag{7.25}$$

so that

$$\sin \xi_L \approx \frac{z_{i-1,j} - z_{i,j}}{\Delta x}\,. \tag{7.26}$$

Likewise, the net contribution of the force pulling to the right is

$$F_R \sin \xi_R = T\, \Delta y \sin \xi_R. \tag{7.27}$$

Now, ξ_R is the angle between the line joining the centers of element (i,j) and element (i+1,j) and the horizontal, or

$$\sin \xi_R = \frac{z_{i+1,j} - z_{i,j}}{\sqrt{\Delta x^2 + (z_{i+1,j} - z_{i,j})^2}} \approx \frac{z_{i+1,j} - z_{i,j}}{\Delta x}\,. \tag{7.28}$$

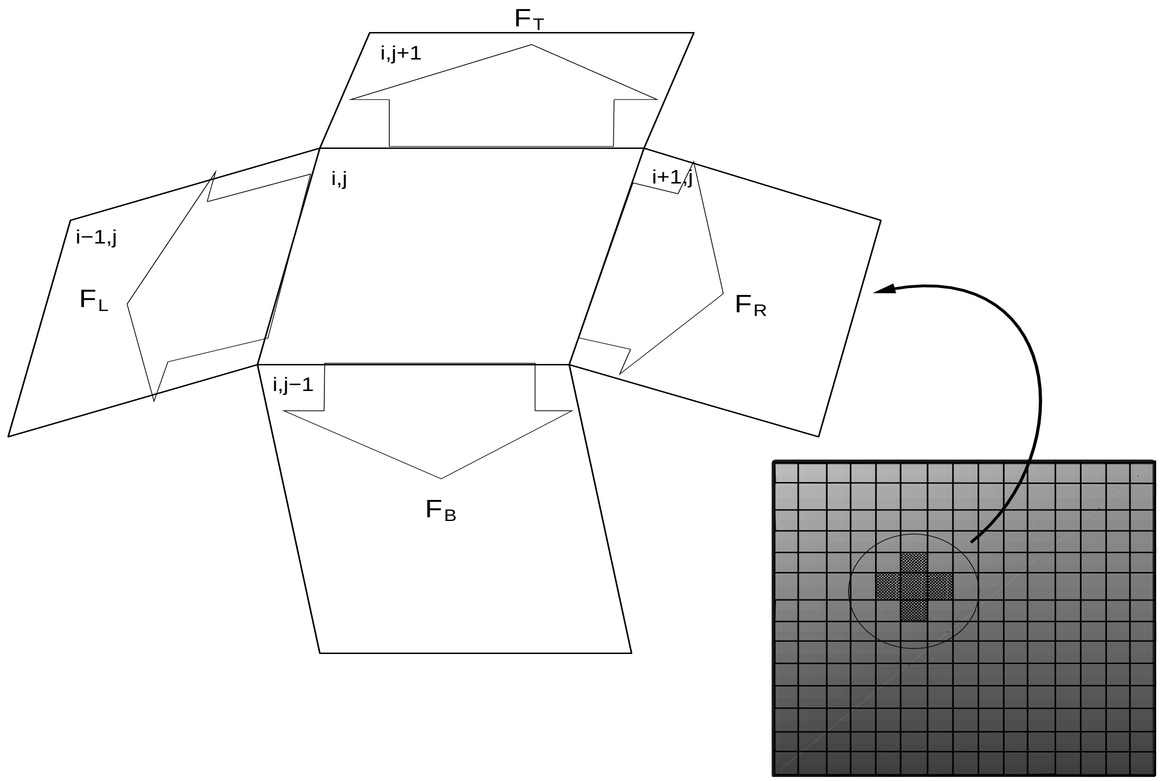

Figure 7.8 *This shows one of the surface elements from a stretched membrane (figure 7.7) which has undergone a small displacement in the z−direction. This surface element is tilted by an angle $\theta_{i,j}$ with respect to the x−axis and $\phi_{i,j}$ with respect to the y−axis. Both of these angles are small.*

The force pulling on the top edge of the zone contributes a net force of

$$F_T \sin \xi_T = T\,\Delta x \sin \xi_T. \tag{7.29}$$

Now the factor of Δx is required because the tension acts along the width of the top edge (Δx) . The angle ξ_T is the angle formed by the line joining the centers of elements (i,j+1) and element (i,j) with the horizontal, thus

$$\sin \xi_T = \frac{z_{i,j+1} - z_{i,j}}{\sqrt{\Delta y^2 + (z_{i,j+1} - z_{i,j})^2}} \approx \frac{z_{i,j+1} - z_{i,j}}{\Delta y}\,. \tag{7.30}$$

Similarly, the net contribution from the force pulling on the bottom edge of the zone is

$$F_B \sin \xi_B = T\,\Delta x \sin \xi_B, \tag{7.31}$$

where ξ_B is the angle between the line joining the centers of elements (i,j) and element (i,j-1) with the horizontal, giving

$$\sin \xi_B = \frac{z_{i,j-1} - z_{i,j}}{\sqrt{\Delta y^2 + (z_{i,j-1} - z_{i,j})^2}} \approx \frac{z_{i,j-1} - z_{i,j}}{\Delta y}\,. \tag{7.32}$$

These four contributions may be summed to produce

$$F_z = T\Delta y\left[\frac{z_{i+1,j} - z_{i,j}}{\Delta x} - \frac{z_{i-1,j} - z_{i,j}}{\Delta x}\right] + T\Delta x\left[\frac{z_{i,j+1} - z_{i,j}}{\Delta y} - \frac{z_{i,j-1} - z_{i,j}}{\Delta y}\right]. \tag{7.33}$$

Newton’s second law implies

$$a_z = \frac{F_z}{m_{i,j}}\ , \tag{7.34}$$

where a_z is the acceleration in the z-direction, and $m_{i,j}$ is the mass of zone i, j. Thus

$$a_z = \frac{T}{m_{i,j}}\,\Delta y\left[\frac{z_{i+1,j} - z_{i,j}}{\Delta x} - \frac{z_{i-1,j} - z_{i,j}}{\Delta x}\right] + \frac{T}{m_{i,j}}\,\Delta x\left[\frac{z_{i,j+1} - z_{i,j}}{\Delta y} - \frac{z_{i,j-1} - z_{i,j}}{\Delta y}\right]. \tag{7.35}$$

comparing this equation with eqn (7.14) implies that the same numerical algorithm designed to simulate the two dimensional array of springs described above will also model a membrane when the point masses are equated with the individual element masses and the forces acting on them in the x- and y-directions are equated with $T\,\Delta y$ and $T\Delta x$, resp.

The mass of an element is

$$m_{i,j} = \sigma \Delta x\, \Delta y \tag{7.36}$$

Substituting eqn (7.36) into eqn (7.35) produces

$$a_z = \frac{T}{\sigma}\left\{\frac{1}{\Delta x}\left[\frac{z_{i+1,j} - z_{i,j}}{\Delta x} - \frac{z_{i-1,j} - z_{i,j}}{\Delta x}\right] + \frac{1}{\Delta y}\left[\frac{z_{i,j+1} - z_{i,j}}{\Delta y} - \frac{z_{i,j-1} - z_{i,j}}{\Delta y}\right]\right\}. \tag{7.37}$$

Those familiar with the Calculus, will recognize this as the two dimensional wave equation

$$\frac{\partial^2 z}{\partial^2 t} = \frac{T}{\sigma}\left[\frac{\partial^2 z}{\partial^2 x} + \frac{\partial^2 z}{\partial^2 y}\right], \tag{7.38}$$

cf. equations (7.37) and (25). Thus any program simulating vibrating membranes in this manner also simulates light waves in two dimensions if it is assumed that

$$\frac{c}{\eta} = \sqrt{\frac{T}{\sigma}}\ , \tag{7.39}$$

where c is the speed of light in vacuum, and η is the index of refraction of the material through which the light wave is traveling.