c. Numerical Examples

Traveling Waves

This first example is intended to familiarize the reader with the operation of the program. The program's setup options should be used to create a triangular displacement in y but with zero velocity, selecting method 3, as shown in figure 5.5. When the program is started, the displacement will
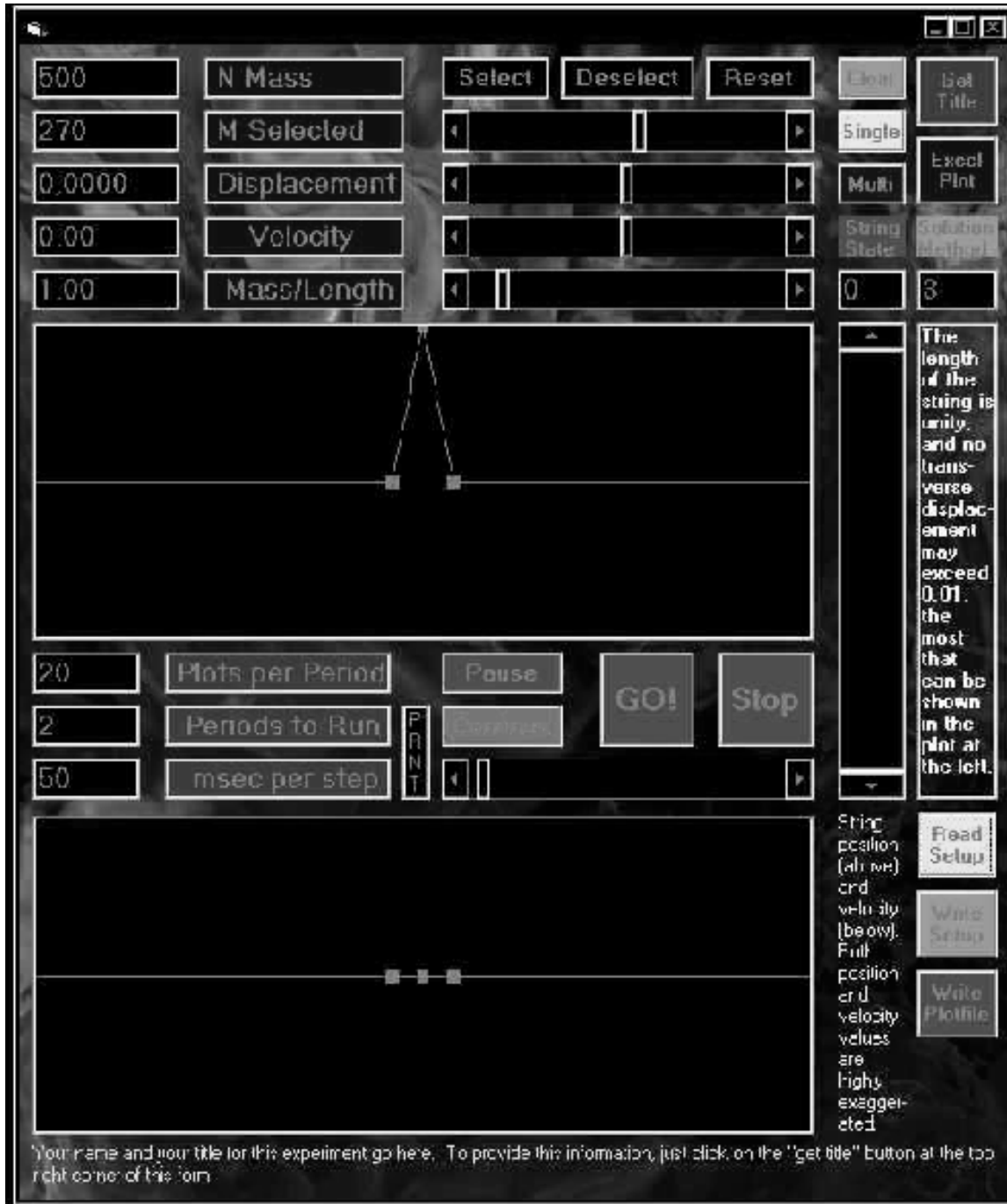

Figure 5.5 *This shows initial conditions consisting of a triangular transverse displacement with no initial velocity. This is the first numerical example described in section 5.c.*

dissolve into two traveling waveforms; one traveling leftward, and one traveling rightward (figure 5.6). This is a general result, as the eager reader can verify with differing initial conditions. The triangular
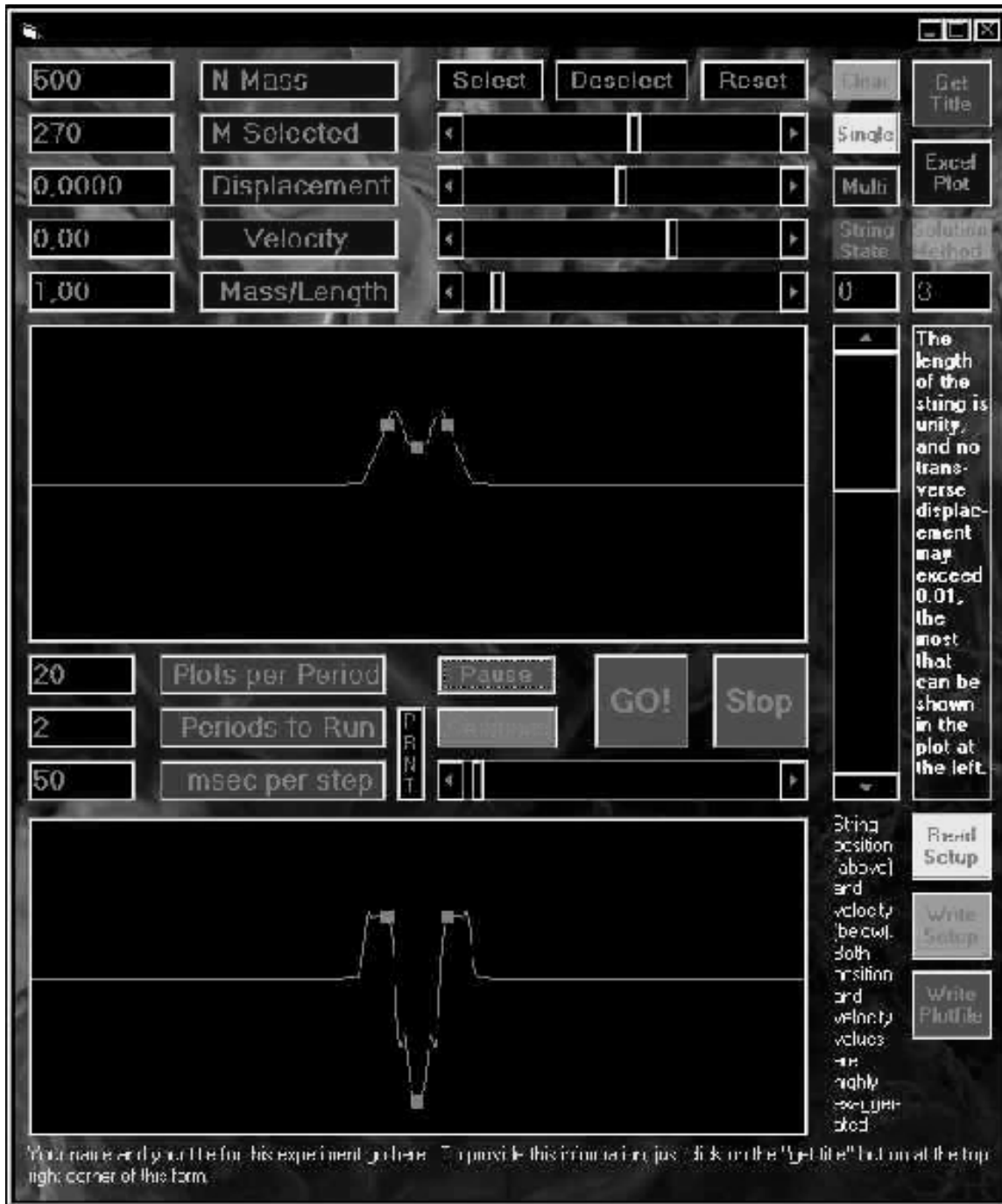
Figure 5.6 *This shows conditions shortly after starting the program. The initial conditions which consisted of a triangular transverse displacement with no initial velocity dissolves into two triangluar waveforms moving in opposite directions.*

waveform will be used here as it is particularly amenable to geometrical analysis.

<center>Traveling waveform</center>

A traveling waveform is a disturbance which changes position without changing shape. When restricted to one dimension, as in this context of a string, the spatial displacement may be described at time $t = 0$ as a function of only one spatial variable. Using x as that variable, this implies

$$y = f(x). \tag{5.23}$$

Then a traveling wave moving to the left (decreasing x) is written

$$y = f(x + v\,\Delta t). \tag{5.24}$$

Here, the velocity of the wave form along the string is represented by v and is assumed constant. The elapsed time, Δt, since the beginning, t^0, is

$$\Delta = t - t^0. \tag{5.25}$$

Equation. (5.24) expresses that after a time interval Δt, the point on the waveform which was initially at $x = 0$ is now located at $x = -v\,\Delta t$, as illustrated in figure 5.7. It follows that a rightward (in the
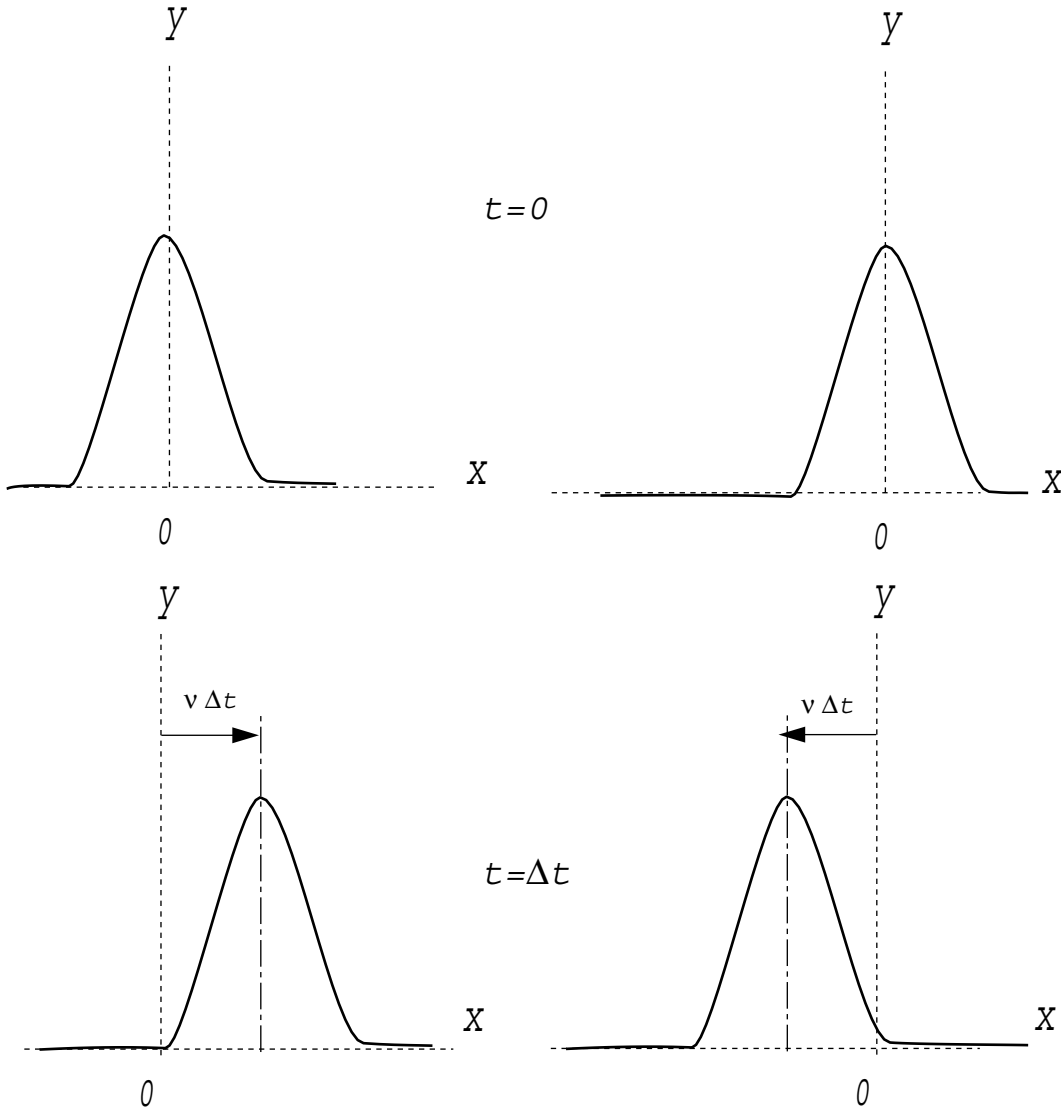

Figure 5.7 *At the initial time, t=0, the wave consists of a single pulse. At a later time, t=Δt, the pulse has traveled to the right a distance vΔt for a rightward moving traveling wave (left column) or a distance vΔt to the left for a leftward moving traveling wave (right column). The velocity v is the speed of sound in the string.*

direction of increasing x) moving waveform is written

$$y = f(x - v\,\Delta t). \tag{5.26}$$

The reader may verify that the two wave pulses generated from the initial state discussed above are in fact traveling waveforms with equal, but opposite, velocities. This is illustrated in figure 5.8.
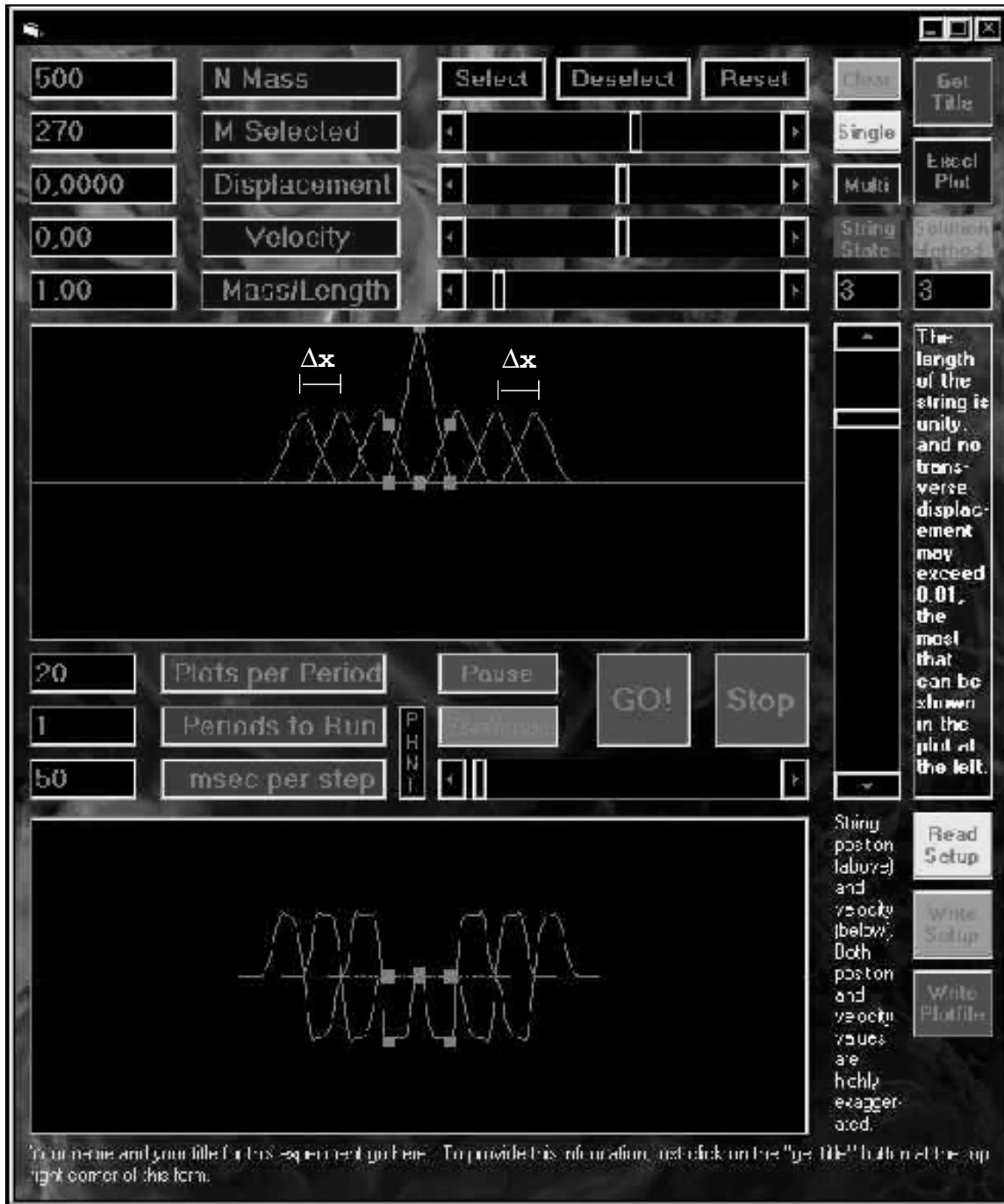

Figure 5.8 *This shows the LCSE string program in multi−display mode. That the velocities of the two waveforms are equal and oppostie may be verified by measuring the distance traveled, Δx, by either wave for a specific time interval. These distances will be equal and opposite.*

The actual value of the velocity may be found using the definition of velocity from eqn. (2.6) repeated here:

$$v = \frac{\Delta(\text{position})}{\Delta(\text{time})} .$$

$$(2.6)$$

This velocity is often referred to as the sound speed of the medium, as sound is just another, but familiar, wave phenomenon. For strings, the speed of sound is

$$c = \sqrt{\frac{T}{\lambda}} \qquad (5.27)$$

where T is the tension in the string, with units of a force, and λ is the linear density, with units of mass per *length*. It is possible to verify that eqn. (5.27) is the proper formulation for the wave speed on a string by using the program to measure the wave speeds for various tensions and densities.

Creating an isolated traveling waveform.

In order to determine the initial conditions for a traveling wave, study carefully the motion of the wave traveling to the right under the conditions suggested at the beginning of this section. It is crucial to realize that no portion of the string moves in the direction of wave motion, the longitudinal direction. Rather, all string motion is perpendicular (transverse). This is obvious for the case of a string with its ends fixed, as the reader may easily verify for him- or herself. It is also true for waves in other materials, such as water.

Now, rather than concentrate on the wave, use the program to consider what happens to a specific element of the string (as if a point on the string had been marked by paint). The element remains at rest (FIGURE 5.9a) until the moment the waveform reaches that point (FIGURE 5.9b) at which point it begins to move upward, implying a positive y-velocity (FIGURE 5.9-c). From the geometry of the waveform (FIGURE 5.10), it can be determined that the upward displacement of the string element (Δy) can be related to the distance by which the wave has overrun the element (Δx):

$$\Delta y = \Delta x \tan(\theta), \qquad (5.28)$$

where $\tan(\theta)$ is a constant depending upon the exact shape of the waveform. For the symmetric waveform used here,

$$\tan(\theta) = \frac{H}{L}, \qquad (5.29)$$

where H is the amplitude of the wave and L is half of its width. But eqn. (2.6) above implies

$$\Delta x = c \, \Delta t. \qquad (5.30)$$

This leads to

$$\Delta y = c \, \Delta t \tan(\theta). \qquad (5.31)$$

Thus the transverse velocity of the string element is

$$v_y = \frac{\Delta y}{\Delta t} = c \, \tan(\theta) = \frac{cH}{L}, \qquad (5.32)$$

until the apex of the waveform passes (fig 5.9d), at which point the string element reverses direction and moves downward (fig 5.9e). Similar geometrical arguments to the above can be used to show that the velocity is now

$$v_y = -\frac{cH}{L}. \qquad (5.33)$$

Those familiar with the calculus may begin with the definition of the transverse velocity, eqn. (2.6):

$$v_y = \frac{\Delta y}{\Delta t}. \qquad (2.6)$$

Using eqn. (5.30) produces

$$v_y = c \, \frac{\Delta y}{\Delta x}. \qquad (5.34)$$

In the limit $\Delta x \to 0$, this becomes

$$v_y = c \, \frac{\partial y}{\partial x}, \qquad (5.35)$$

which can be a useful result when initializing problems in general. For example, to create a traveling sine
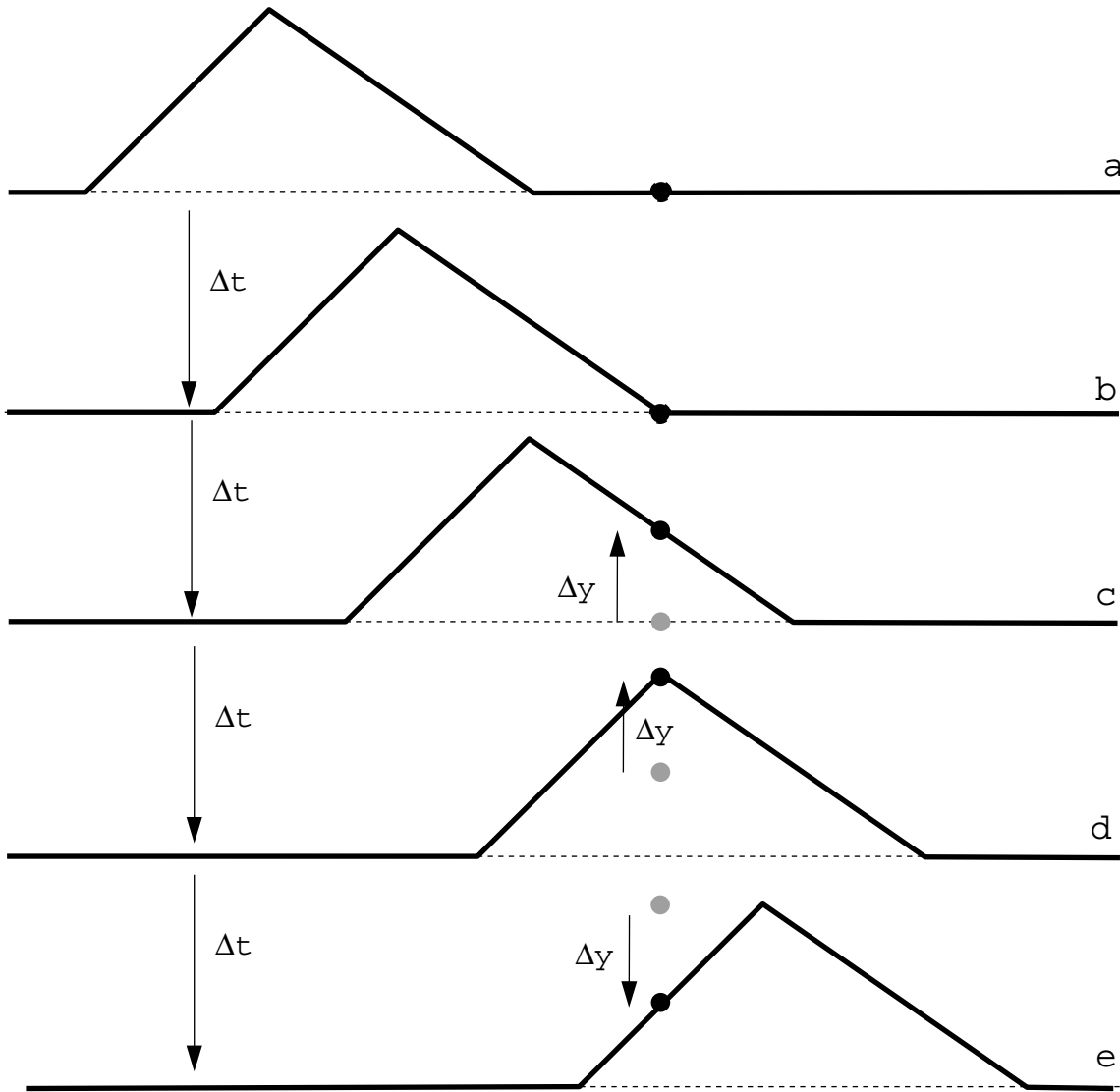
Figure 5.9 *This schematic illustrates what happens to a specific element of the string (indicated here by the dot) when a triangular waveform passes. Depicted hare are five snapshots separated in time by a uniform Δt. (Time increases downward.) The element remains at rest (a) until the moment the waveform reaches that point (b), at which point it begins to move upward, implying a positive y−velocity (c), until the apex of the waveform passes (d). At that point the string element reverses direction and moves downward (e).*

wave with the initial transverse displacement

$$y(x) = A \sin(\alpha x), \tag{5.36}$$

requires initial velocities of

$$v_y(x) = A c \alpha \cos(\alpha x). \tag{5.37}$$

To summarize: a leftward triangular traveling wave form is defined by a spatial displacement
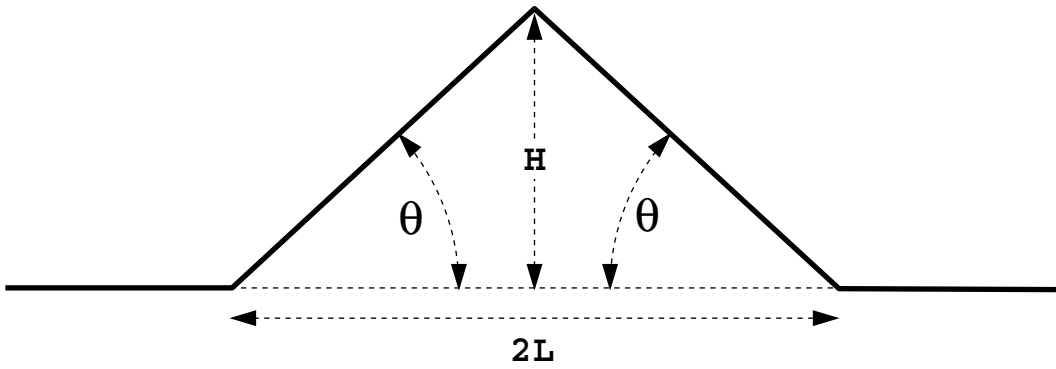
Figure 5.10 *This depicts the geometry of a symmetrical triangular wave pulse. The maximum amplitude is H and the full width of the pulse is 2L. Thus from basic trigonometry tan(θ) = H/L.*

$$y(x) = \begin{cases} 0, & \text{if } |x| \le L, \\ H(1+\dfrac{x}{L}), & \text{if } 0 \le |x| \le L \end{cases} \quad (5.38a)$$

and a transverse velocity distribution of

$$v_y(x) = \begin{cases} 0, & \text{if } |x| > L, \\ \dfrac{cH}{L}, & \text{if } 0 \le x \le L, \\ -\dfrac{cH}{L}, & \text{if } -L \le x \le 0, \end{cases} \quad . \quad (5.38b)$$

It is left as an exercise to the reader to determine that a rightward traveling wave is defined by the spatial displacement

$$y(x) = \begin{cases} 0, & \text{if } |x| \le L \\ H(1+\dfrac{x}{L}), & \text{if } 0 \le |x| \le L \end{cases} \quad (5.39a)$$

and a transverse velocity distribution of

$$v_y(x) = \begin{cases} 0, & \text{if } |x| > L \\ -\dfrac{cH}{L}, & \text{if } 0 \le x \le L \\ \dfrac{cH}{L}, & \text{if } -L \le x \le 0 \end{cases} \quad . \quad (5.39b)$$

The reader is strongly encouraged to use the program to verify these equations do in fact describe traveling waves. These equations also hold when the sign of *H* changes (that is, for a dip). Note that this also changes the sign on the velocities.

wave superposition

The program will now be used to examine the principle of *superposition*: which states that when two or more waves are present, they act independently of one another and the resulting waveform is the sum of the these individual components. Begin by using the setup options to create two traveling triangular waveforms as in fig. 5.11.
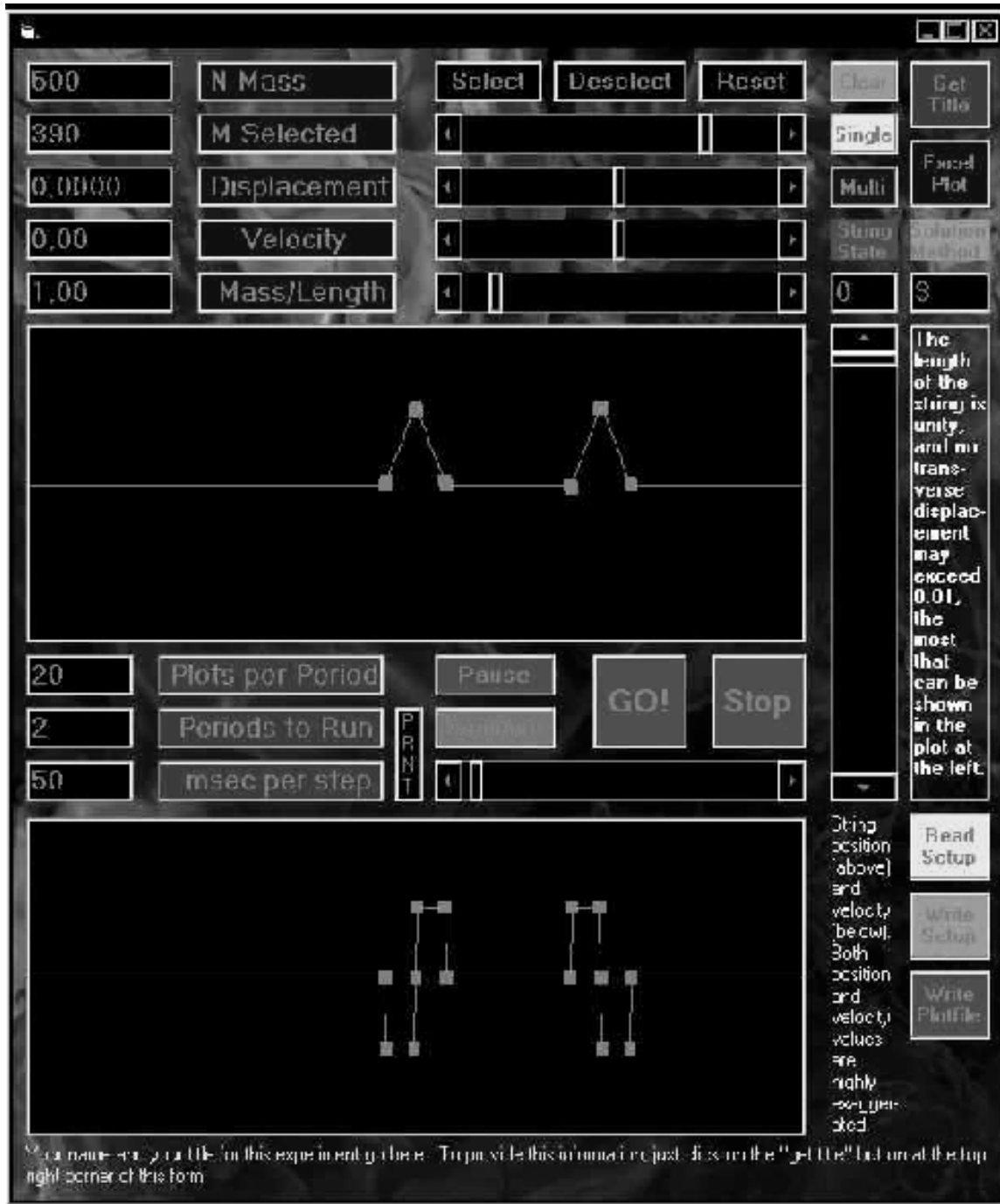

Figure 5.11 *This shows the initial conditions to create two triangluar waveforms travelling towards each other. Both waveforms have a positive amplitude.*

The leftward moving waveform should be to the right of the rightward moving waveform so that they will move toward one another when the program is started. Each should have the same geometrical shape with their velocities differing only in sign, as necessary to determine their directions. Run the program and halt it when the two waveforms overlap. Note that the amplitude of the resulting displacement is twice that of

either of the two traveling waves and that the waveform is simply the sum of the two traveling waves. Also note that if the two velocity distributions are added, they sum to zero when the overlap is exact. This is the condition described at the beginning of this section, showing that the results of that first example are a general result of wave superposition. Continuing the program from this point shows this single triangular displacement dissolving back into the two component waveforms, another consequence of superposition. Because the two waves act independently, they can pass through one another without changing their individual characteristics.

The next example uses a similar pair of waves, but with amplitudes mirrored as shown in figure 5.12. (Note that the velocities are also mirrored to again produce waves traveling towards one another.) Now when the two waves to approach and pass through one another, their amplitudes will cancel. Note that when their positions exactly overlap, no spatial disturbance is visible, but if the program is allowed to continue, each wave emerges unscathed traveling in its own initial direction. At that instant of exact overlap, the spatial displacements of the two waves cancelled, but not their velocities. This can be verified by trying the resultant velocity distribution without any initial transverse displacement, as the initial conditions.

Energy of a wave

This example uses wave superposition to provide a simple way of determining the total energy of the two waves. Since there is no displacement during the moment of exact overlap, the energy of the waves must consist entirely of kinetic energy. (Recall that while the transverse displacements summed to zero, the transverse velocities did not.)

For a particle of mass m and velocity v its kinetic energy is

$$\text{K.E.} = \frac{1}{2} mv^2. \tag{5.40}$$

Here, during the moment of exact overlap, there exist two uniform sections of string, each with mass λL moving with velocities $\pm cH/L$, thus the kinetic energy and hence total energy is

$$\frac{\lambda c^2 H^2}{L}. \tag{5.41}$$

By symmetry, each individual waveform must have half of this value or

$$E = \frac{\lambda c^2 H^2}{2L}. \tag{5.42}$$

Reflection and transmission at an interface

The program may be used to simulate a wire formed by joining two lengths of differing linear densities. Consider the case when the linear density of the left half is four times that of the right half. From eqn. (5.27), the sound speed in the right will be twice that in the left. A rightward traveling wave in the left half of the wire as in figure 5.13 will divide into two pulses, one reflected and one transmitted, figure 5.14.

Note that the width of the transmitted pulse is twice that of the initial pulse. This can be understood geometrically by realizing that when the leading edge of the pulse enters the less dense material it must immediately move at the faster wave speed, while the trailing edge, which is still in the less dense material, must continue at the slower speed. This results in a stretching of the wave form. To get the exact ratios, consider that a waveform traveling at a velocity c requires a time Δt to travel a distance Δx of

$$\Delta t = \frac{\Delta x}{c}. \tag{5.43}$$

Using the widths of the two waveforms as Δx produces

$$\Delta t_1 = \frac{2L_1}{c_1}, \tag{5.44a}$$

and

$$\Delta t_2 = \frac{2L_2}{c_2}, \tag{5.44b}$$

where the subscripts identify the two differing densities. These two times must be equal, since the point immediately to the left of the interface is contiguous with the point immediately to its right,
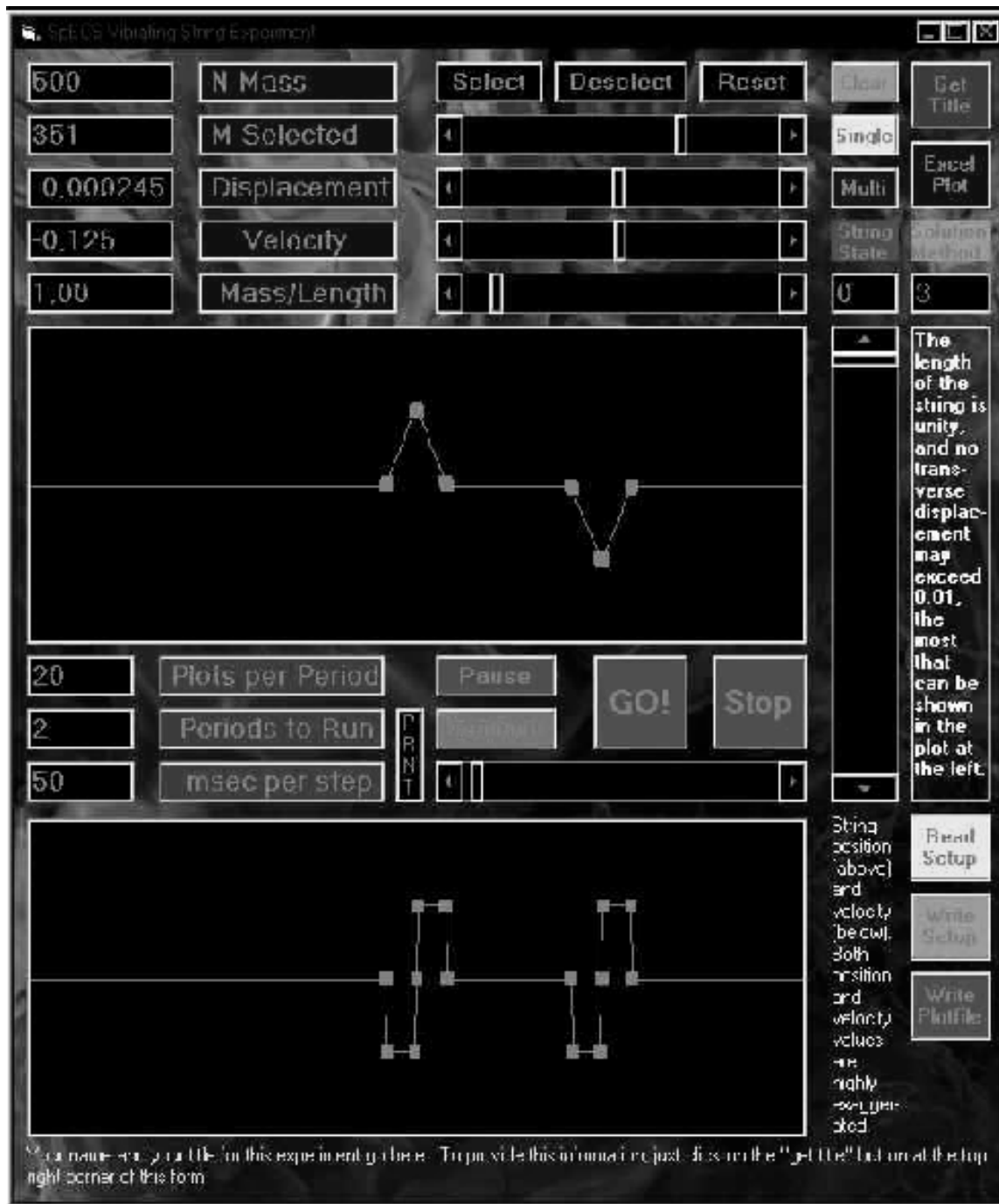
Figure 5.12 *This shows the initial conditions to create two triangluar waveforms travelling towards each other. One waveform hase a positive amplitude while the other has a negative amplitude. Compare and contrast these initial conditions with those of figure 5.11, in which both waveforms have positive amplitudes.*

$$\frac{2L_1}{c_1} = \frac{2L_2}{c_2}, \tag{5.45a}$$

or

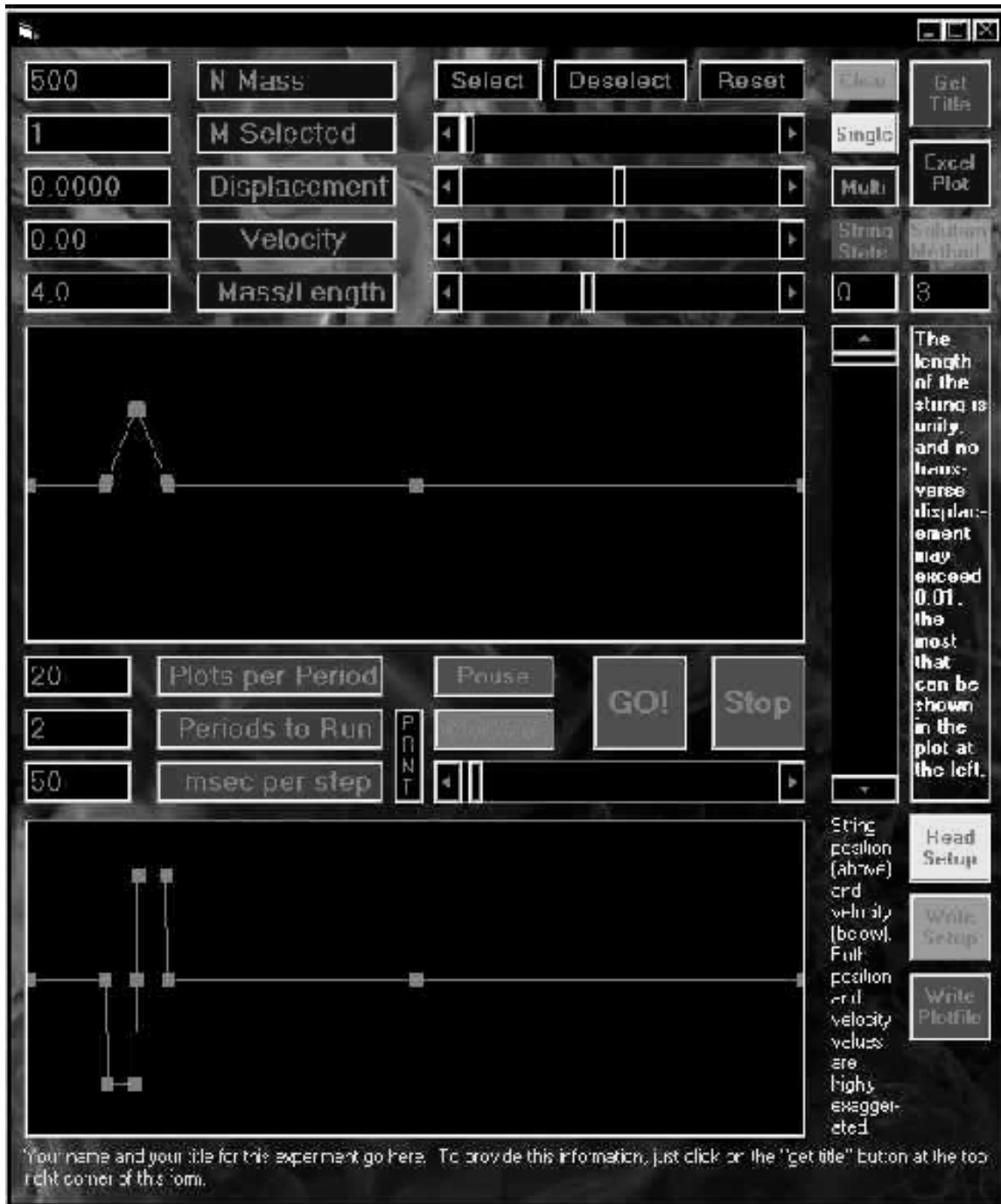$$L_1 = 2L_2\,\frac{c_1}{c_2}. \tag{5.45b}$$

Figure 5.13 *These are the initial conditions for the example of wave reflection at an interface as described in the text. Two lengths of differing linear densities are joined at the center. The density of the left half is four times that of the right. A rightward travelling wave is established in the left half of the string.*

Of course, the reflected pulse has the same width because it remains in the same material.

In order to determine the amplitudes of the reflected H_r and transmitted H_t waves in terms of the amplitude of the incident (original) wave, H_i, two pieces of information will be required. (There are, after all, two unknowns.) These will come from conservation of energy and the continuity of the string.
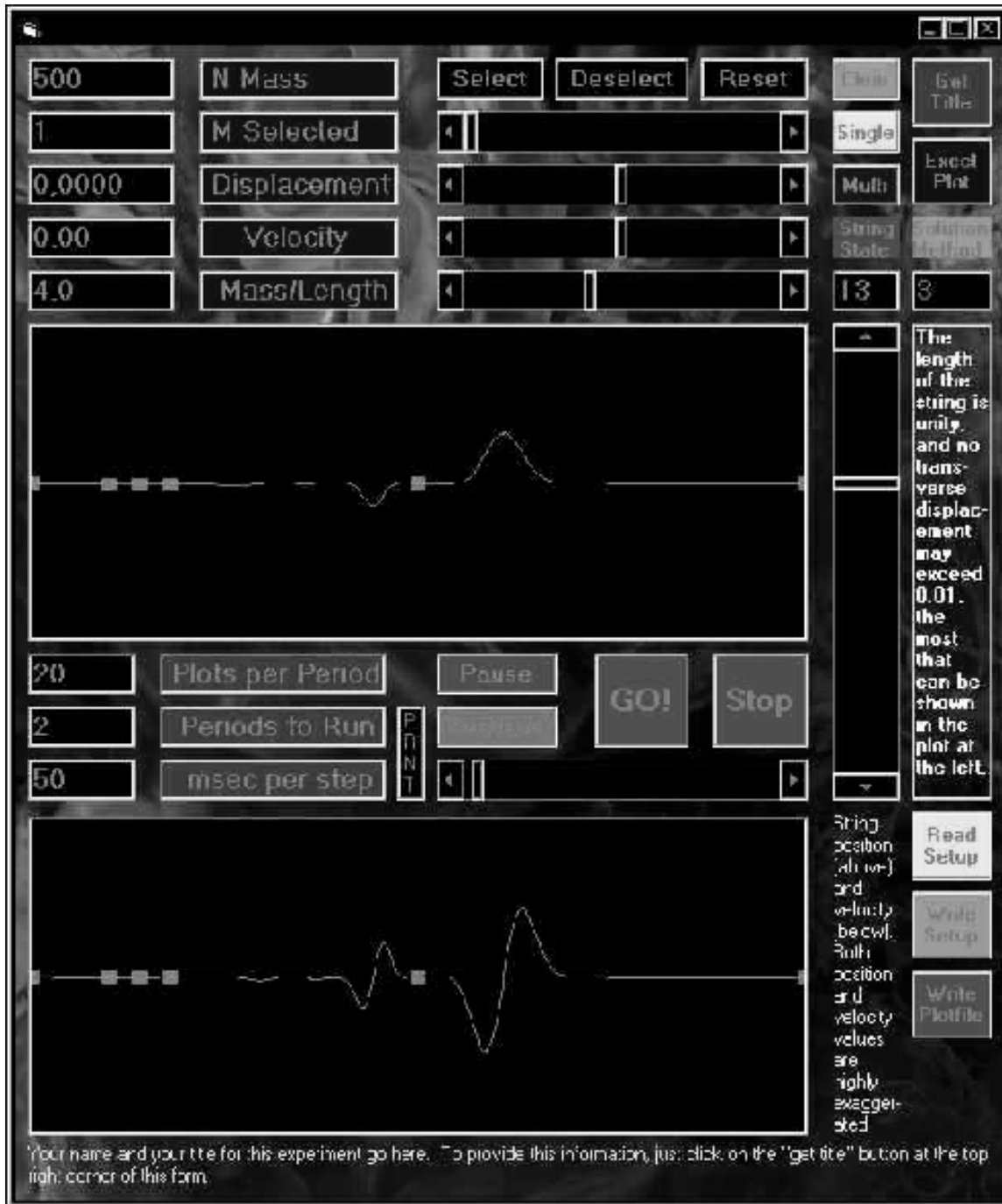
Figure 5.14   *Two lengths of differing linear densities are joined at the center. The density of the left half is four times that of the right. A rightward travelling wave which was established in the left half of the string has struck the density interface. At the interface, this wave split into two travelling waves: a reflected wave travelling left, and a transmitted wave traveling right.*

Conservation of energy requires that the energy of the sum of the reflected and transmitted waves must equal the energy of the incident wave. Using eqn (5.42) to equate the energies on either side of the interface produces

$$\frac{\lambda_1 c_1^2 H_i^2}{2L_1} + \frac{\lambda_1 c_1^2 H_r^2}{2L_1} = \frac{\lambda_2 c_2^2 H_t^2}{2L_2} \qquad (5.46a)$$

or

$$H_t^2 = \sqrt{\frac{\lambda_1}{\lambda_2}}\left(H_i^2 - H_r^2\right)$$ (5.46b)

In order to prevent a discontinuity developing at the interface, the transverse velocity immediately to the left must equal the velocity immediately to the right of the interface. The velocity on the left is the sum of the incident and reflected waves (when they overlap) thus

$$\frac{c_1 H_i}{L_1} + \frac{c_1 H_r}{L_1} = \frac{c_2 H_t}{L_2}$$ 47a)

which simplifies to

$$H_t = H_i + H_r.$$ (5.47b)

Eqns (5.46b and (5.47b) may be solved for H_r and H_t:

$$H_r = \frac{\sqrt{\lambda_1} - \sqrt{\lambda_2}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}} H_i$$ (5.48)

and

$$H_t = \frac{2\sqrt{\lambda_1}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}} H_i.$$ (5.49)

These equations may be verified by the string program using various combinations of linear densities. It should be noted from equation 67 that when a waveform reflects from an interface for which the string beyond is *denser,* the amplitude of the reflected wave is reversed.

The energies of the reflected and transmitted waveforms may now be found. Using eqn (5.48) and simplifying gives

$$E_t = \frac{4\sqrt{\lambda_1 \lambda_2}}{\left(\sqrt{\lambda_1} + \sqrt{\lambda_1}\right)^2} E_i$$ (5.50)

for the transmitted wave and

$$E_r = \frac{\left(\sqrt{\lambda_1} - \sqrt{\lambda_1}\right)^2}{\left(\sqrt{\lambda_1} + \sqrt{\lambda_1}\right)^2} E_i$$ (5.51)

for the reflected wave.

These lead to the coefficient of reflection, which is the ratio of reflected energy to incident energy,

$$R = \frac{\left(\sqrt{\lambda_1} - \sqrt{\lambda_1}\right)^2}{\left(\sqrt{\lambda_1} + \sqrt{\lambda_1}\right)^2}$$ (5.52)

and the coefficient of transmission,

$$T = \frac{4\sqrt{\lambda_1 \lambda_2}}{\left(\sqrt{\lambda_1} + \sqrt{\lambda_1}\right)^2}$$ (5.53)

When $\lambda_1 = \lambda_2$, that is, when the string is continuous, these coefficients reduce to $R \rightarrow 0$ and $T \rightarrow 1$ as expected.

Now consider the effects of adding a short length of an intermediary density at the interface, forming a wire made of three segments. Several things will be observable. First, when the pulse strikes the first interface, the intensity of the reflected wave is reduced from the previous example because the difference in

densities at the interface has also been reduced. The intensity of the transmitted wave is greater for the same reason. The transmitted wave continues until it reaches the second interface, at which point it splits again into a reflected and transmitted wave. The final transmitted wave is slightly stronger than in the previous example. If the second reflected wave is studied, it will be seen to bounce back and forth between the two interfaces, splitting into a reflected and transmitted wave at each bounce, albeit with decreasing amplitude. This is analogous to reverberation in a large auditorium, or the internal reflections often seen in thick glass plates. The sign amplitude of the pulse trapped between the two interfaces reverses on every other reflection, when it reflects from a boundary where the string beyond has a greater density.

This idea of increasing the transmission of a wave at an interface by inserting an intermediate material can be extended by adding more pieces of intermediate densities. The logical extension of this process is to match the two densities with a short segment of linearly and continuously increasing density. This is analogous to *impedance matching* in electronic circuits.

6. Numerical Artifacts

The most common numerical artifact encountered with the string program results from discontinuities in the initial conditions. Figure 6.1 shows an initial displacement consisting of a discontinuity in the transverse direction. When run with this initial condition, the program produces two traveling waves, as expected, but the edges of the waves exhibit "ringing," which was hot expected (figure 6.2). The overall properties are correct (two oppositely traveling waves) but the small details are not (the ringing). This should not be surprising, since the initial conditions violate the assumptions used in deriving the program, particularly that the string was smooth and continuous inherent in applying eqn. (2.12). What these initial conditions represent - one element of the string displaced many times its diameter (recall that the y-axis is magnified in the string display) - is in reality a broken string. Trying to create such conditions in a real string (without breaking it) would require the transverse displacement to occur across a few string elements. As figure 6.3 shows, spreading the discontinuity over just a few elements (here six) dramatically reduces the numerical artifacts.
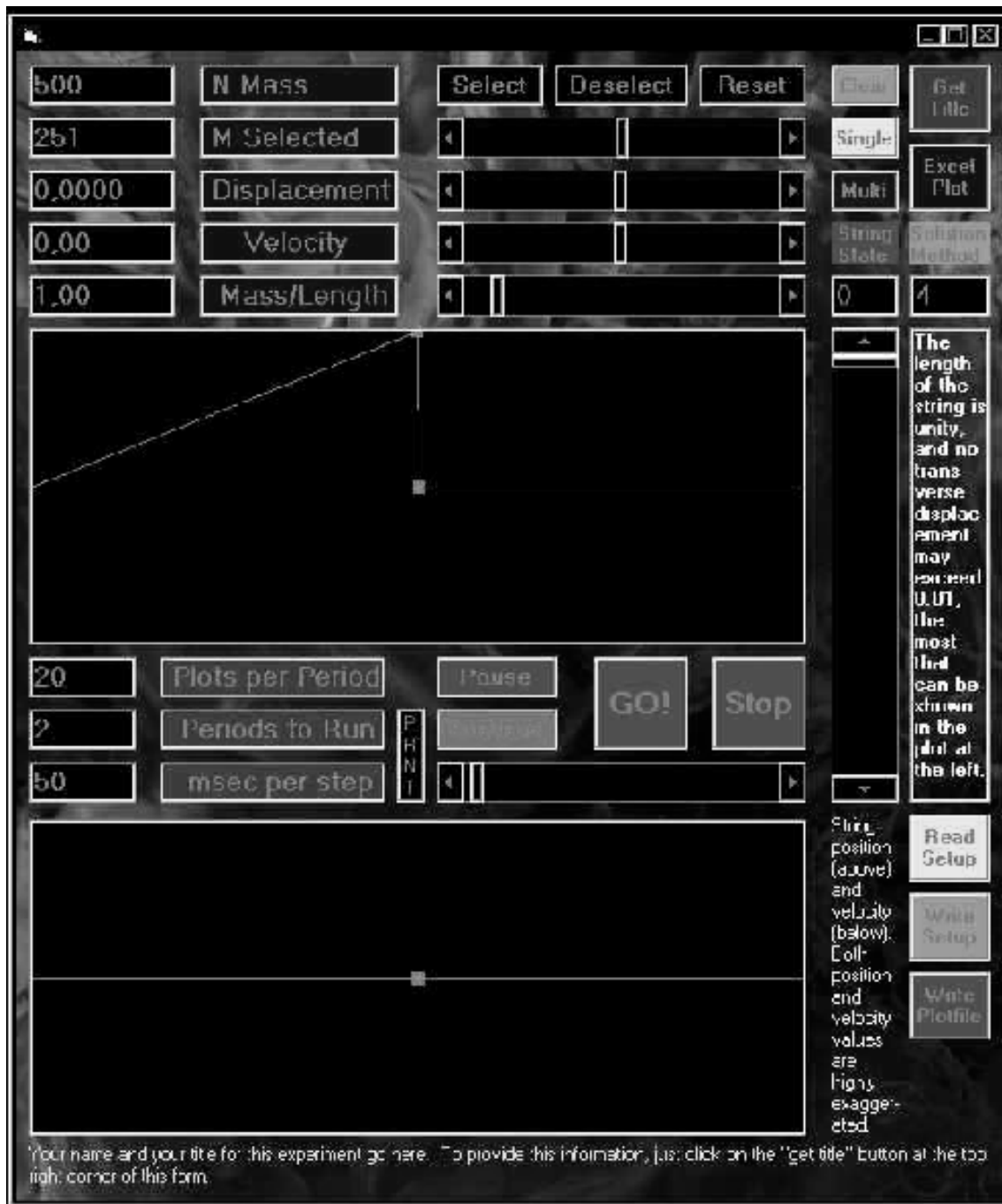
Figure 6.1 *This shows initial conditions consisting of a discontinuity in the transverse displacement. The displacement (which is large compared to the assumed diameter of the string) occurs over only one string element. In reality, this would correspond to a broken string.*
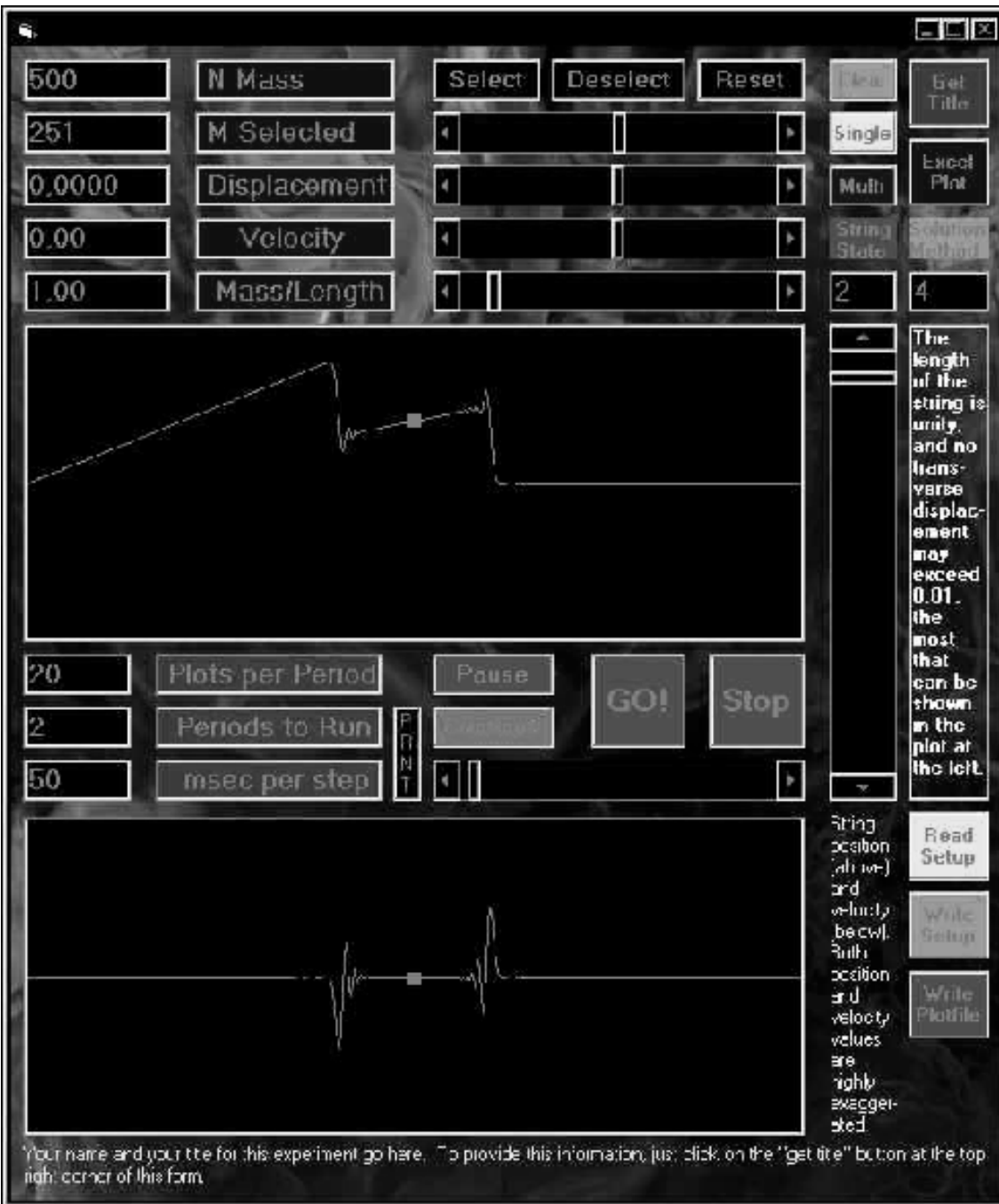
Figure 6.2 *The discontinuity of figure 6.1 produces two waves travelling in oposite directions. The leading edge of each wave exhibits ''ringing'' which is a numerical artifact. This arises because the initial conditions for this problem violate the assumptions used in deriving the program, particularly that the string was smooth and continuous.*
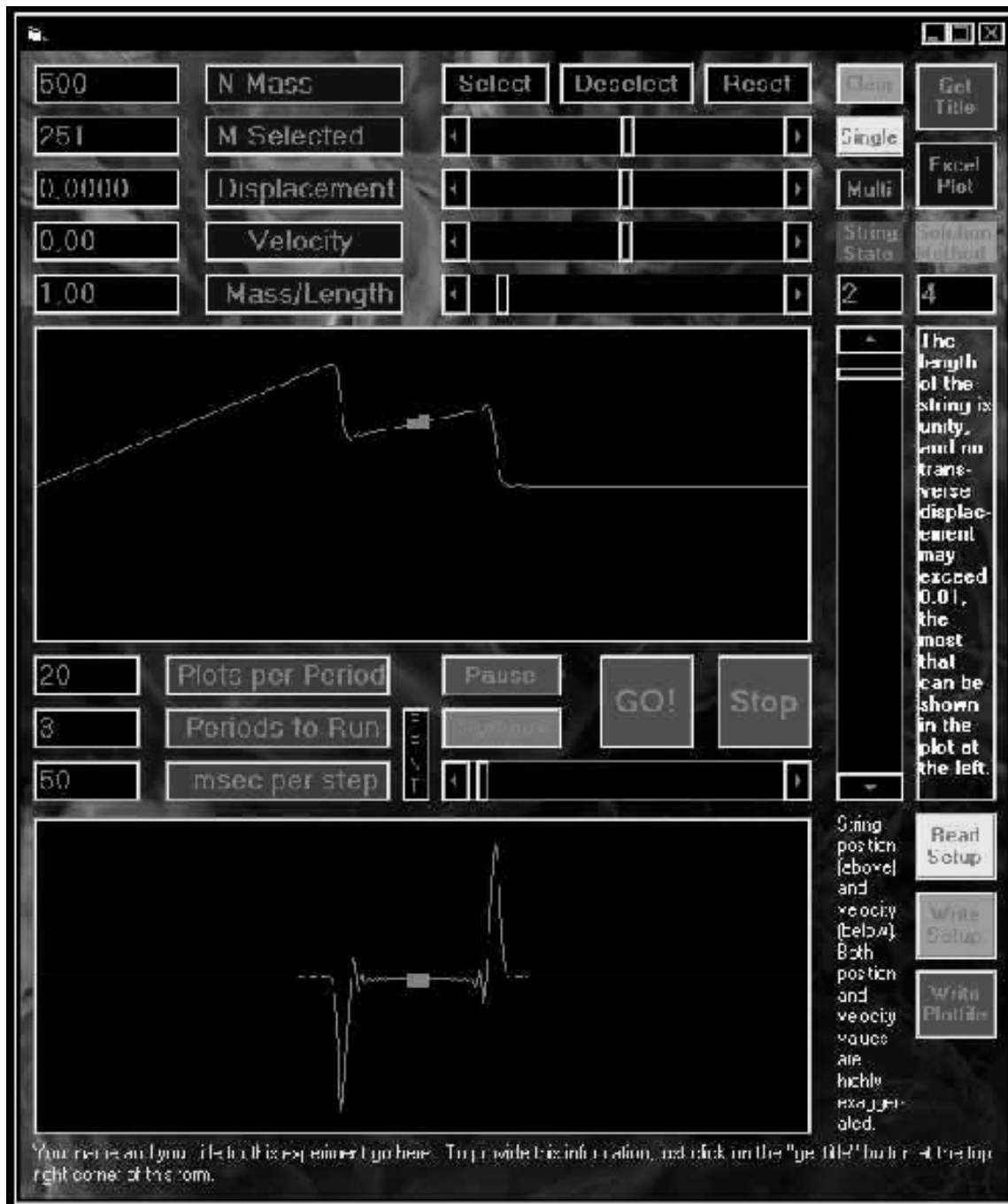
Figure 6.3   *When the discontinuity of figure 6.1 is spread over a few elements, the numerical artifacts are greatly reduced. As the initial discontinuity is spread over more and more elements, the initial conditions match the assumptions used in deriving the program more and more closely.*