3-D Simulations of Turbulent Compressible Convection

David H. Porter[1] and Paul R. Woodward[1]

Astronomy Department, University of Minnesota, Minneapolis, MN 55455

Author to which all correspondence is to be directed: David Porter, (612) 624–1817

ABSTRACT

Three–dimensional simulations of turbulent and fully compressible thermal convection in deep atmospheres are presented and analyzed in terms of velocity power spectra, mixing–length theory, and production of vorticity. Density contrasts across these convective layers are typically around 11. The fluid model is that of an ideal gas with a constant thermal conductivity. The Piecewise-Parabolic Method (PPM), with thermal conductivity added in, is used to solve the fluid equations of motion. No explicit viscosity is included, and the low numerical viscosity of PPM leads to a very low effective Prandtl number and very high effective Rayleigh number. Mesh resolutions range as high as 512×512×256, and the corresponding effective large–scale Rayleigh numbers range as high as 3.3×10^{12}.

Compressional effects lead to intensely turbulent downflow lanes and relatively laminar updrafts, especially near the top boundary. The enstrophy contrast between downflows and upflows increases with mesh resolution (and hence with decreasing viscosity), and ranges as high as a factor of 30 in our highest resolution model. Vorticity is everywhere preferentially aligned with the principal direction of strain associated with the large–scale circulation. Near the top boundary, the strain field associated with the largest scale of convection dominates, which leads to a 2-D horizontal network of vortex tubes. For the same reason, both

[1]Laboratory for Computational Science and Engineering, University of Minnesota, Minneapolis, MN 55455

the upper portions of the downflow lanes and the lower portions of the updrafts contain many strong vertical vortex tubes with helicities of random sign. The horizontal vortex tubes near the very top of the downflow lanes tend to come in counter rotating pairs, with one on each side of the downflow lane. Therefore, similar to observations of the sun, there are upflows along each side of the prominent downflows in our simulations. Mach numbers in these convective layers are largest in the upper, more diffuse region. There they range as high as 0.8, which significantly modifies the pressure and gravitational force balance from that which would apply under static conditions. This effect is incorporated into our mixing–length analysis of the simulation data.

Subject headings: convection, stars:interiors, hydrodynamics, turbulence, videotapes

1. Introduction

In this paper we present a series of computer simulations of compressible convection in a three-dimensional stratified atmosphere which is confined between two frictionless plates. The purpose of this work is to determine the nature of such convective motions when the viscosity of the gas is very low. Our ultimate purpose is to study the effects of heat and material transport in stellar convection zones. As yet we have resisted the temptation to introduce many features of stellar convection into our models which would make them at the same time more realistic and more difficult to interpret. In this approach we have followed the early example of Hurlburt et. al. 1984, and it was indeed these investigators who originally suggested, more than a decade ago, that we undertake this work. This paper presents a natural extension of our earlier 2-D simulations of compressible convection (Woodward et. al. 1987, Porter and Woodward 1988, Porter and Woodward 1994) to three dimensions. Our earlier work on 3-D compressible convection (Porter and Woodward 1989, Porter et. al. 1990, Porter et. al. 1991, Woodward et. al. 1995, Porter and Woodward 1996) is extended here to simulations with higher grid resolution. Simulations of compressible convection by other investigators complement the work presented here by investigating local area models with wider aspect ratios for the simulation volume (Malagoli et. al. 1990, Cattaneo et. al. 1991, and Bogdan et. al. 1993), local area models with rotation (Nordlund and Stein 1993, Brummell et. al. 1995), models of convection with penetration into stable layers above or below (Singh et. al. 1994,Hurlburt et. al. 1994, Singh et. al. 1995), models of global convection (Glatzmaier and Toomre 1995), or by adding significant effects not considered here, such as ionization, opacity tables, and/or magnetic fields (Sofia and Chan 1984, Xiong 1989, Nordlund 1985, Cattaneo 1992, Nordlund et. al. 1992).

We simplify the problem of solar convection by neglecting the following aspects of the problem: (1) the presence of magnetic fields, (2) the variation of the thermal conductivity due to radiation diffusion with the temperature of the gas, (3) the changing state of ionization of the gas, (4) the rotation of the system, (5) the curvature of the convection zone due to the spherical geometry of the sun, (6) the presence of a penetrable stable gaseous layer below the convection zone, and (7) the existence of a free stellar surface from which radiation escapes. With all these simplifications, one might well ask if any aspect of significance to solar convection remains in our simulations. We feel that indeed our study addresses the nature of convection which takes place over several density and pressure scale heights and in a gas of very small viscosity. By focusing our attention clearly upon these effects we are able to make definitive statements about them and to quantify the accuracy of those statements. Were we to undertake simulations including many of the neglected effects, we feel that it would be most difficult to disentangle these many effects in order to understand the resulting fluid behavior. Our approach will be instead to add these effects one by one into our simulations so that their impact on the results is made clear.

In this work we concentrate on trying to resolve the hydrodynamic flows that form in 3-D thermal convection in deep atmospheres. We seek to find which small scale structures are produced and persist in thermal convection, and we wish to find the effects that the presence of these small scale structures have on the larger scale flow. Perhaps the most common treatment of the influence of fluid turbulence on the structure of stellar convective envelopes is to use an adaptation of Mixing–Length Theory (MLT) in order to relate the total energy flux to the radial temperature gradient (Vitense 1953 and Bohm–Vitense 1958). Three–dimensional numerical models of stellar convection are free of many of the assumptions needed solve a 1-D (i.e., radially averaged) description of the problem and also provide complete flow information which can be used to test the framework and results of MLT. The assumptions underlying MLT can be tested quantitatively in terms of correlations of the fluctuating quantities of temperature, density, and velocity. A wide variety of these kind of statistical measures have been made for various models of convection in deep atmospheres in previous work (Chan and Sofia 1989, Singh and Chan 1993, Chan and Sofia 1996, Kim et. al. 1996) where nonuniform meshes and a sub–grid scale turbulence model were employed (Chan and Sofia 1986) at modest mesh resolutions. The numerical simulations presented here use uniform meshs at higher resolution and can be used to check the validity and robustness of these previous results as well as test MLT in new parameter regimes.

Another issue involves the form of the turbulent velocity spectrum produced by convection in a deep atmosphere, which can feed back on the transport of heat and effect the vertical thermodynamic profile of the atmosphere. Such considerations are used in the full spectrum of turbulence (FST) modification of mixing–length theory (Canuto and Massitelli 1991, Canuto and Massitelli 1992, Canuto et.al. 1996), which uses adaptations of the Eddy Damped Quasi–Normal Markovian approximation (EDQNM) models (Orzag 1977, Canuto et. al. 1991) and the Direct Interaction Approximation (DIA) models (Kraichnan 1964, Hartke et.al. 1998) to thermally driven convection. These models of turbulence tend to generate a range of scales in which energy power spectra scale as $k^{-5/3}$, characteristic of an Kolmogorov–Obukhov inertial range. However, the presence of multiple pressure scale heights which span a wide range of sizes can lead to broad band buoyancy driving and vitiate the assumption of an inertial range.

In the work presented here, we use a high resolution difference scheme, the Piecewise-Parabolic Method, PPM (Woodward and Colella 1984, Colella and Woodward 1984, Woodward 1986), to solve the Euler equations of inviscid gas dynamics with very high accuracy. We have modified the PPM scheme by introducing an explicit thermal conduction to drive the convection. Concentrating our attention and all our grid resolution on a small section of the deep, convectively unstable atmosphere, we examine the nature of an intensely turbulent convection cell.

Our computations, which exhibit a wealth of complicated and nonlinear structures, cannot be properly analyzed without the examination of enormous quantities of data. Much of this data is summarized in the form of color movies of vorticity presented on the video which accompanies this article. A wide variety of hydrodynamic quantities from our earlier simulations of compressible convection have been presented as movies (see for example Woodward 1988, Porter and Woodward 1989, Porter et. al. 1990, Porter et. al. 1991). In both 2- and 3-D these movies provide us with an intuitive understanding that allows us to form relevant models for the individual constituents, as well as the overall behavior, of our convective systems. In section 2 we specify the convective systems that we study in this work, and define parameters that are useful in describing the global properties of our simulations. In section 3 we present the results of our simulations of various convective systems computed with a range of computational mesh resolutions. And in section 4 we discuss our conclusions.

2. Definition of the Problem

We model these convective flows in terms of an ideal, polytropic, inviscid gas. We choose as dynamical variables the density ρ, pressure P, and velocity $\mathbf{u}$. The pressure can be related to the density and an internal energy ϵ via $P = (\gamma - 1)\rho\epsilon$, where γ is the adiabatic index and has the value of 5/3. We have a constant heat capacity at constant volume c_v, which relates the internal energy to the temperature by $\epsilon = c_v T$. The dynamical variables are functions of spatial coordinates (x, y, z) and time t. We impose a uniform gravitational field, with a constant acceleration due to gravity given by g pointed in the $-\hat{\mathbf{z}}$ direction. We model radiative transfer in terms of a constant coefficient of heat conduction κ. The equations of motion can be written as

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \quad , \qquad (2.1)$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla P}{\rho} - g\hat{\mathbf{z}} \quad , \qquad (2.2)$$

$$\partial_t(\rho E) + \nabla \cdot (\mathbf{u}\rho E) = -\nabla \cdot (\mathbf{u}P - \kappa \nabla T) \quad , \qquad (2.3)$$

where the total energy density per unit mass is given by

$$E = \frac{1}{2}\mathbf{u}^2 + \epsilon + gz \quad . \qquad (2.4)$$

We choose units so that the mean density ρ_0, the depth of the layer $D \equiv z_{top} - z_{bot}$, the acceleration due to gravity g, and the heat capacity at constant volume c_v are all unity. Boundary conditions are periodic in the two horizontal directions, which are the coordinate directions x and y in all cases except for run c11 (see Table 1 and section 3.1 below). We impose impenetrable and frictionless walls for the boundaries at top and bottom. We impose a heat flux F_T, along the lower boundary, and a temperature, T_{top}, along the upper boundary. The imposed vertical heat flux is spatially uniform everywhere along the lower boundary, constant in time and the same for all of the models presented here (see Table 1).

Imposing a uniform temperature along the top boundary which is constant in time and has the same value for all of these models would generate several problems for the purpose of generating a series of models in convective equilibrium which could be directly compared. First, given the extremely small values of the coefficient of heat conduction in these systems, the time for thermal relaxation is far longer than any other time in these models. For example, given a fixed temperature along the top boundary and $\kappa = 10^{-4}$ it takes hundreds of time units for even one e-folding relaxation time of the total energy in the system. With $\kappa = 10^{-5}$ it takes thousands of time units for the total energy in such a system to relax. By contrast, the time for the largest scales to overturn is several time units, and the time for the largest scale convection cells to form is a few hundred time units. The second problem is that in systems which T_{top} is fixed and which span several pressure scale heights the vertical profiles of density and pressure depend sensitively on the temperature structure near the top boundary. However, the thermal boundary layers along the top boundary, as well as the turbulent vertical energy transport, are underresolved in the lower resolution models presented here : section 3.2 shows in detail how the lower pressure scale heights, and increasingly more of the uppermost pressure scale heights, are resolved as mesh resolution is increased. Convection is efficient in the lower pressure scale heights of our models where it drives the temperature profile to be nearly adiabatic. Hence, the mean vertical profiles of density and pressure will be nearly identical in the lower pressure scale heights, even for different values of κ, F_T, and F_R/F_T, provided that the total mass, energy and depth of the systems are the same. For the purpose of comparing systems with different mesh resolution, κ, F_T, and F_R/F_T it is better to keep the total energy and total mass of the systems the same than to keep T_{top} the same.

For these reasons, we specify a temperature which is spatially uniform along the top boundary at each time, but is allowed to change with time so as to maintain a nearly constant total energy in each model. The energy flux which exits through the top boundary is entirely due to thermal conduction and is $F_top = 2\kappa(T_{inside} - T_{top})/\Delta x$, where T_{inside} is the mean temperature in the layer of computational cells just below the top boundary, and Δx is the width of one zone. The total energy in the system would be constant, to numerical roundoff, over a time step if $F_top = F_T$, which leads to $T_{top} = T_{inside} - \Delta x/(2\kappa)$. In order to filter out temporal fluctuations we use a running time average for T_{inside} of the form $< T_{inside} > (t) = \int_0^t e^{\frac{-t-\tau}{\delta t}} T_{inside}(\tau) d\tau$. In all of the models presented here the filter time, δt, is set to unity. With this prescription the total energy does not change appreciably and quickly settles to a nearly constant value (see Figure 6 and related text at the end of section 3.1).

The lowest resolution models in each series (e.g., c11, c21, and c31, see Table 1) all start from a slightly superadiabatic hydrostatic vertical profile with small velocity perturbations. These lowest resolution models ($N_z = 32$) each take a few hundred time units for the total kinetic energy to reach its equilibrium value, and are allowed to run for 1000 time units to allow each model to come into complete convective equilibrium. Each of the higher resolution models, starting with c22 and c32 as listed in Table 1, start from a mesh refined version of the final state of the preceding member of the series. These higher resolution models come into convective equilibrium in about two turn over times, with the mean vertical energy flux at each depth being equal to the imposed flux, F_T, along the bottom (see Figures 11c and 11d and related text in section 3.2) and the velocity spectra fully developed and in equilibrium (see Figures 4 and 5 and related text in section 3.2).

The numerical method used is the Piecewise Parabolic Method, or PPM (Colella and Woodward 1984, Woodward and Colella 1984, Woodward 1986). It can be regarded as a method of numerically modeling the small–scale dissipation, just as large–eddy simulations model the unresolved small–scales with explicit transport coefficients. A comparison of results obtained with both a Navier–Stokes solver and PPM for the problem of homogeneous turbulence in two dimensions is reported in Porter et. al. 1990 and Porter et. al. 1992a. Convergence tests of PPM for three dimensional turbulence is reported in Porter et. al. 1994.

Table 1 lists the simulations reported here. Eight runs are reported in all, each run is identified by the run name in column 1. All of the simulations have aspect ratios of 2 by 2 by 1, with the short dimension being in the vertical direction. Each mesh resolution (N_x, N_y, N_z), column 2 of table 1, is chosen to make cubical computational zones, which minimizes anisotropies of the numerical diffusion. Each run is carried through sufficient model time, column 3, to allow turbulence to develop fully and global quantities, like the total kinetic, heat, and gravitational energies, to relax.

The imposed heat flux along the lower boundary is also the total energy flux across the layer, F_T. Convection efficiently mixes most of the layer in all of these models, so the radiative flux, F_R, at most depths is directly related to the coefficient of heat conduction κ (shown in column 5), and the adiabatic temperature gradient, $[\partial T/\partial z]_{ad} = -\beta_{ad} = -g/(c_v\gamma)$. F_R/F_T is the fraction of the total energy flux carried by radiation in each model and is shown in column 6. It is instructive to relate the flux that must be carried by convection, $F_T - F_R$, to the Mach number,

$$M_o = (2(F_T - F_R)/\rho_o)^{1/3}/c_o \quad , \qquad (2.5)$$

corresponding to the entire convective flux being carried by the kinetic energy flux, where ρ_o and c_o are the density and speed of sound at the base of the convective layer. The Mach number M_o for each run is shown in column 4.

We use the measures of the numerical viscosity of PPM reported in Porter and Woodward 1994 to estimate the effective large scale Rayleigh, Ra, and Prandtl, Pr, numbers shown in columns 7 and 8 respectively. The effective shear viscosity due to the numerical dissipation of PPM on an isolated shear wave goes as

$$\nu_s = \frac{A_s}{4\pi^2}\left[\frac{\lambda}{\Delta x}\right]^3 u_0\lambda \qquad (2.6)$$

where λ is the wavelength of the velocity wave with amplitude u_0 being numerically damped on a mesh composed of uniform zones of size Δx. The coefficient A_s depends on the advective Courant number. The advective Courant number is about 0.14 in all the simulations reported here, which corresponds to $A_s = 10$. To estimate the large scale values of Ra and Pr, we use a wavelength of unity (i.e., the depth of the layer). The amplitude of flow velocities on large scales is about $u_0 = 0.03$ for runs c11, c21, c22, and c23, and is about $u_0 = 0.05$ for runs c31, c32, c33, and c34. The effective large–scale Prandtl number for these systems

$$\mathbf{Pr} = \frac{\nu_s}{\kappa} \quad , \qquad (2.7)$$

goes as N_z^{-3}. For the Rayleigh number, we need the coefficient of thermal expansion $Q = -[\partial ln\rho/\partial lnT]_P$, the adverse temperature gradient, β, of the imposed heat flux relative to the adverse adiabatic temperature gradient, β_{ad}, and an estimate of the mean temperature in the system. For a γ–law gas $Q = 1$, and $T = 0.41$ at mid depth on average in all of the systems reported here. The adiabatic temperature gradient is $\beta_{ad} = 0.6$ for these $\gamma = 5/3$ systems, and the imposed temperature gradient along the lower boundary is $\beta_B = 0.75$ for runs c11, c21, c22, and c23, and is $\beta_B = 7.5$ for runs c31, c32, c33, and c34. The effective

Table 1

Summary of the Numerical Simulations.

Run	$N_x \times N_y \times N_z$	Time	M_o	κ	F_R/F_T	Ra	Pr
c11	$92 \times 46 \times 32$	1000	0.021	10^{-4}	0.80	1.577×10^{10}	2.319×10^{-3}
c21	$64 \times 64 \times 32$	1000	0.021	10^{-4}	0.80	1.577×10^{10}	2.319×10^{-3}
c22	$128 \times 128 \times 64$	300	0.021	10^{-4}	0.80	1.261×10^{11}	2.899×10^{-4}
c23	$256 \times 256 \times 128$	125	0.021	10^{-4}	0.80	1.010×10^{12}	3.623×10^{-5}
c31	$64 \times 64 \times 32$	1000	0.035	10^{-5}	0.08	4.354×10^{12}	3.865×10^{-2}
c32	$128 \times 128 \times 64$	200	0.035	10^{-5}	0.08	3.483×10^{13}	4.831×10^{-3}
c33	$256 \times 256 \times 128$	200	0.035	10^{-5}	0.08	2.787×10^{14}	6.039×10^{-4}
c34	$512 \times 512 \times 256$	46	0.035	10^{-5}	0.08	2.229×10^{15}	7.549×10^{-5}

large–scale Rayleigh number is

$$\mathbf{Ra} = \frac{gQl^4}{T\kappa\nu_s}(\beta - \beta_{ad}) \quad , \qquad (2.8)$$

and goes as N_z^3 for each series of runs.

3. Results

3.1. Large–Scale Circulation

The convection simulations considered here (see Table 1) all possess the same modest horizontal to vertical aspect ratio, x:y:z = 2:2:1, and have nearly the same vertical stratification, typically with 4.5 pressure scale heights across the layer and a density contrast of $\chi_\rho = 11$. In every case these factors lead to very similar large scale flows, characterized by a network of downflow lanes near the top boundary which merge together to form two intersecting and perpendicular downflow lanes at mid depths, which then merge into one large downflow plume near the lower boundary.

Horizontal structures in run c23 (see table 1) are shown in gray scale Figures 1a-1f. Figures 1a, 1b, and 1c, show the vertical component of velocity at $z = 0.25$, 0.5, 0.9375 respectively, the coordinate z is the height measured from the lower boundary in units of the layer thickness. Lighter shades of gray indicate upward velocity, darker shades represent downflows. Note the network of downflow lanes near the upper boundary (Figure 1a). As depth increases, the number of downflow lanes decreases, and the size and turbulence of the downflow lanes increases. At $z = 0.5$ (Figure 1b) there are two large downflow lanes, which intersect. At $z = 0.25$ (Figure 1c) the regions of downflow have concentrated into a few large and turbulent plumes. Fluctuations in the temperature at the same three depths are shown in Figures 1d through 1f. Negative fluctuations in the temperature are seen to be fairly well correlated with downflows. However, the fluctuations of vertical velocity which are apparent in the broad upflow regions seen in Figures 1b and 1c have no corresponding fluctuations in the temperature (Figures 1e and 1f).

Similarly, horizontal structures in run c34 are shown in gray scale in Figures 2a-2f. The horizontal sections are taken at $z = 0.25$, 0.5, 0.9375 These figures show the same general pattern of a network of downflow lanes near the top boundary (Figures 2a & 2d), which converge to form to large downflow lanes, which are at right angles to each other, at mid depth (Figures 2b & 2e), and a large downflow plume in the lower pressure scale height (Figures 2c & 2f). Run c34 is representative of higher Rayleigh and Peclet numbers, but nearly the same Prandtl number, as run c23. The higher Peclet number of c34 leads to greater levels of thermal turbulence, especially in upflows (Figures 2d-2f) than in c23 (Figures 1d-1f). However, in both sets of runs and at all resolutions both velocity and temperature fluctuations are the strongest in downflow lanes.

At mid depths there is a tendency for regions of downflow to be organized into two well defined lanes which not only are at right angles to each other but also tend to be aligned with the axes of the simulation. Here, the downflow lanes might be aligned with the computational mesh directions, which would correspond to a numerical artifact. Alternatively, the downflow lanes might be aligned with the fundamental periodic directions of the model, which could be an effect of the modest aspect ratio of these runs. In the two series of runs c20 to c23 and c31 to c34 the fundamental periodic directions are exactly the mesh directions, so there is an ambiguity as to which potential cause is responsible for the preferential direction of downflow lanes at mid depths. As a test, we constructed a model where the fundamental periodic directions were at 45^o to the mesh directions. This was implemented by choosing a mesh which was half as long in the y direction as the x direction, imposing standard periodic boundaries in the x direction, but for the boundaries in the y direction copying zones along 45^o diagonals. For example, to generate the value for the field F at a point (x, y, z) which lies just below the lower y boundary we used

$$F(x, y, z) = F(x + L, y + L, z) \qquad (3.1)$$

where L is the y dimension or half the x dimension. In such a mesh the shortest (or fundamental) periodic directions are at 45^o to the x or y direction. Run c11, with mesh dimensions of $(N_z,N_y,N_z) = (92,46,32)$ was run with such a boundary, and with all other parameters made to match those of run c21, which was run on a $64\times64\times32$ mesh (see table 1). Figures 3a-3f show a comparison of these two runs. Figures 3a, 3b, and 3c show the vertical velocity from run c21 in horizontal sections at z=0.5, 0.6875, 0.875 respectively. Figures 3d, 3e, and 3f show the vertical velocity from run c11 at the same three depths, here the volume is replicated a factor of 2 in the y direction, with the appropriate shift in x, to show the full convective pattern. As Figures 3a-3f show, the downflow lanes at mid depths align themselves with the fundamental periodic directions, and not the mesh directions, even in these extremely low resolution runs. While these downflow lanes gradually move, and occasionally break up and reform, Figures 3a-3f show typical configurations of downflow lanes in these models. The modest aspect ratio chosen in these simulations restricts the large scale shape of the flow. This is a price we are willing to pay in order to maximize the mesh resolution across substructures, such as downflow lanes, in order to study the interaction of fluid turbulence and convection.

The major downflow lanes forming a cross which is aligned with the principal periodic directions is a robust feature for our choice of aspect ratio, and is present over a wide range of mesh resolutions. In fact, the large scale flow in all of these convection models has the same form, in some detail, independent of mesh resolution. Power spectra of the vertical velocity give us one measure of the flow field over a range of scales. In these vertically stratified models we anticipate that there will be systematic changes with depth, so it is appropriate to take the power spectra on horizontal cuts. Figure 4a shows 2–D Fourier power spectra of the vertical velocity field taken on the plane at mid–layer ($z = 0.5$) in the system. Here we are plotting the average spectral energy per mode, $A(u_z)$, as opposed to the total energy in a spherical shell, or circular annulus, in wave number space. The three curves show the power spectra for the series of runs c21, c22, and c23, which are all identical except for their mesh resolutions, which range from 64×64×32 to 256×256×128. Wavenumber k has units of inverse distance and is consistent with the unit of distance being the depth of the layer. Since the aspect ratio is 2x2x1 the minimum wavenumber is $k_{min} = \pi$. In Figure 4b, we show the same plots for the runs c31, c32, c33, and c34 which range in mesh resolution from 64×64×32 to 512×512×256. In each series of runs the vertical velocity spectra agree at each k to within the level of statistical fluctuations, independent of mesh resolution, from $k = k_{min}$ to $k = k_{max}/16$, where k_{max} is the Nyquist wavenumber of the mesh in each case. All seven spectra shown in Figures 4a and 4b have their peak amplitudes at $k = k_{min}$ and decrease strongly with k, which is consistent with the flow being dominated by a single convection cell which spans the depth and breadth of the simulation region, as discussed above. Numerical dissipation is seen to have a strong effect up to wavelengths of about $10\Delta x$, where the curves drop off sharply. A reasonable power–law fit to $A(u_z)$ at long wavelength is $k^{-8/3}$, which is the power law corresponding to a Kolmogorov inertial range spectrum in the $A(u_z)$ vs. k plane. A $k^{-8/3}$ power-law is shown in Figures 4a and 4b for comparison.

The shapes of the velocity power spectra in the dissipation range are very similar from one resolution to the next. In Figures 4c and 4d we compare the dissipation ranges between the three models c21,c22,c23 (Figure 4c) and the four models c31,c32,c33,c34 (Figure 4d) by scaling the wavenumber by the maximum (i.e., Nyquist) wavenumber k_{max}, scaling the vertical velocity spectrum $A(u_z)$ by its value at k_{max}, and then compensating each spectrum by $(k/k_{max})^{-8/3}$ in order to remove the overall trend for $k < k_{max}/4$. We see that the dissipation ranges ($k > k_{max}/4$ or $\delta < 8\Delta x$) match quite well in both series of runs. Agreement for $k < k_{max}/4$ is also remarkably good. A Kolmogorov inertial range would be horizontal in Figures 4c and 4d.

The slope of these velocity power spectra are close to that expected for an inertial range over more than one decade of size scales in the case of run c34 (see Table 1), which suggests that the flow is primarily driven, in the sense of injection of energy, on only the largest scales. In principle, these flows are driven on scales ranging from the size of the smallest cold drips forming in the thermal boundary layer along the top boundary, which are as small as a few computational mesh cells wide, to the scale of the large downflow lanes, which can span the depth and breadth of the region simulated. The flow is driven by buoyancy forces, which we can measure as a function of scale by examing the Fourier power spectra of the mass density ρ in 2-D horizontal cuts at various heights. Figures 5a and 5b show the total Fourier power of ρ for run c23 at heights $1-1/a$ for a =2, 8, and 64 (Figure 5a) and for run c34 at heights $1-1/a$ for a=2, 8, 32, and 128 (Figure 5b). For both runs the buoyancy forcing, which just goes as ρ, is fairly broad band, with contributions, at some height, for $1 \leq k/k_{min} \leq 32$. The overall circulation in these convection models ensures that velocity perturbations driven at one height are quickly advected to all other heights.

We can directly quantify the strength of buoyancy driving of the kinetic energy by examining the correlation of density fluctuations with vertical velocity ($\delta\rho u_z$) at each height. When averaged over the periodic horizontal extent of our models each Fourier mode of $\delta\rho$ couples only with the same Fourier mode of u_z, hence we can decompose the buoyancy driving of the kinetic energy, $\int \delta\rho u_z dxdy = \int \tilde{\rho}_k \tilde{u}_{z,k} dk_x dk_y$, into contributions from each scale (specified by $|k|$). We find that the peak driving at each depth within the layer occurs at a wavelength roughly equal to the local pressure scale height, which happens to be is comparable to about half the distance to the upper boundary. At each depth, and on scales smaller than those of peak driving, the buoyancy driving of the kinetic energy scales as $k^{-5/3}dk$. Since the bulk of the buoyancy driving occurs in the lowest pressure scale height, which fills the bulk of the volume, the buoyancy driving averaged over the entire volume peaks at $k/k_{min} = 4$, and also scales as $k^{-5/3}dk$ for larger k. Approximately half of the buoyancy driving of the kinetic energy comes from modes $k/k_{min} \leq 5$ and about 92% of the buoyancy driving comes from $k/k_{min} \leq 32$. Pressure gradients also contribute to the driving of the flow. However, pressure fluctuations are relatively small in these models : the power spectrum of pressure fluctuations is 20 time smaller than that of the density fluctuations for all wavelengths. Further, the pressure gradients seem to be completely uncorrelated with the velocity field at all wavelengths, so that the injection of energy into the kinetic energy field from pressure gradients is negligible compared to the contributions from mass fluctuations. We conclude that a Kolmogorov like velocity spectrum (i.e., $k^{-8/3}$ energy per mode in a 2-D slice scaling as $k^{-8/3}$, see Figures 4a to 4d) can be present in a range of scales where there is significant driving of the kinetic energy. For example, in run c34 the forward energy flux at a wavenumber $|k|$, which is due to the total energy input from larger scales, is not constant but increases by almost a factor of two over the range $4 \leq k/k_{min} \leq 20$, while in this same range the velocity spectra scale as $k^{-8/3}$ per mode or equivalently as $k^{-5/3}dk$. Kolmogorov-like velocity coexisting in a range of scale where the forward energy flux increases significantly is also seen in models of decaying, isotropic turbulence (Porter et. al. 1998).

There is an excess (above the $k^{-8/3}$ asymptote at small k), or “bump”, in the velocity spectra for $k_{max}/16 < k < k_{max}/4$. This excess in spectral energy in the near dissipation range is also seen in 3-D numerical models of isotropic decaying turbulence (Porter et. al. 1992a, Porter et. al. 1994, Porter and Woodward 1996), has been identified as a potential k^{-1} range (Porter et. al. 1992b), and has been observed in wind tunnel experiments. As pointed out in Porter et. al. 1994, the dissipation on the smallest scales inhibits the usual kink instability that these vortex tubes would otherwise experience and thus decreases the effectiveness of energy transfer from this near dissipation range to the strong dissipation range. The result is a flattening of the power spectrum for the solenoidal (incompressible) component of the velocity with, in this near dissipation range, the solenoidal velocity power spectra displaying an approximate k^{-1} power law, as first noted in numerical simulations in Porter et. al. 1992b. Reanalysis of the experimental data of Gagne obtained in the wind tunnel of Modane also indicates a k^{-1} behavior in the near dissipation range (Gagne 1987, and She and Jackson 1993) as well as computations in the incompressible case using either filtered data centered on vortex filaments (Jimenez et. al. 1993) or a hyperviscosity code (Borue and Orszag 1994). The amplitude of this bump has been found to be less for driven turbulence (about 20% excess in velocity spectra) than for decaying turbulence (about 250% excess) (Porter et. al. 1995). In these continuously driven convection systems, the excess seems to be about 40%.

As mentioned above, the horizontal scale of convection cells (or downflow lanes) decreases as the upper boundary is approached. This can be seen in the velocity power spectra. Figures 5c and 5d show the average energy per Fourier mode in the vertical velocity taken in 2-D horizontal cuts at heights $1 - 1/a$ for a = 2, 4, 8, 16, 32, and 64 for each of the two runs c23 and c34. The overall trend is for the power spectra at a height $1-1/a$ to be nearly constant up to roughly $k = a/2$. Recall that k is an inverse length in units of the depth of the layer. Hence, at a depth of 1/a below the upper boundary, modes of wavelength $2/a$ and larger are suppressed, which corresponds to the constraint that the cross section of an eddy must be roughly circular.

Some care is taken to ensure that all of the power spectra discussed above, and all of the statistical measures given in sections III.2 and III.3 below, are taken at times after each system has come to convective equilibrium. As discussed in section II, the temperature along the top boundary is allowed to vary so as to keep the total energy nearly constant. Time histories of the total energies (Figures 6a and 6b), mean temperature along the top boundary (Figures 6b and 6c), and mean flux along the top boundary (Figures 6e and 6e) from the four highest resolution models (e.g., c22, c23, c33, and c34 as listed in Table 1) illustrate how quickly these systems come into equilibrium. The equilibrium total energies differ by no more than one part in 5200, which leads to essentially identical vertical profiles of density and pressure in the lower pressure scale heights between all of these models. In each case the mean temperatures along the top boundary, T_{top}, is seen to come to an equilibrium value within a few turnover times, or about 25 time units. The equilibrium values of T_{top} are seen to differ only slightly for the F_R/F_T = 0.8 cases of c22 and c23, but differ substantially for the F_R/F_T = 0.08 cases of c33 and c34. These variations in the temperature along the top are entirely consistent with the constraint that the temperature profiles should be nearly identical in the lowest pressure scale heights (as they are in these models) and that the temperature gradients be different in the upper pressure scale heights (as they are expected to be for different convective fluxs and different mesh resolutions, see section section 3.2). The mean energy flux through the top boundary, F_{top}, shows the effects of temporal fluctuations in the mean temperature just below the top boundary. In each case, within 25 time units of the initial state, F_{top} is seen to oscillate around the energy flux which is imposed along the lower boundary (i.e., F_T) and vary by no more than a few percent from that value, indicative of these systems being in equilibrium. Figures 11c and 11d, in section 3.2 below, show that the total energy fluxs (averaged both over time and horizontal position) for runs c23 and c34 are very close to the imposed flux F_T at every height within the simulation volume, which is an even stronger indicator of convective equilibrium.

3.2. Vertical Profiles and Mixing–Length Theory

The amplitude of convective velocities in these models can be related to the energy flux which we impose along the lower boundary. Figures 7a and 7b show the mean, rms, and extrema of the vertical component of velocity as functions of height from runs c23 and c34 respectively. The mean (solid line) is nearly zero everywhere, the *rms* fluctuations about the mean (error bars) decrease to zero at the impenetrable boundaries, and are otherwise roughly independent of depth away from the boundaries. The peak downflow velocities are seen to be about 3 times larger in magnitude than the peak upflow velocities, which reflects the asymmetry between upflows and downflows. We impose the same heat flux through the lower boundary in these two runs. However, the coefficient of heat conduction used in c34 is ten times lower than that of c23. The vertical energy flux that needs to be carried by convective motions in c34 is much larger than that of c23. Let f_o be the imposed heat flux along the bottom boundary, and $f_r = \kappa g/\gamma$ be the vertical radiative energy flux for an adiabatic atmosphere. The velocity scale, u_f, derived by equating the kinetic energy flux which would result from a vertical velocity of u_f, given the local density ρ, to the convective energy flux $f_o - f_r$, is

$$u_f = (2(f_o - f_r)/\rho)^{1/3} \quad , \qquad (3.2)$$

which is shown as the dashed curves in Figures 7a and 7b. The slight increase of u_f with height is due to the decrease of density with height. We see that the *rms* vertical velocity scales fairly well with u_f between the runs c23 and c34.

The equation of hydrostatic equilibrium is commonly assumed in models of stellar structure, because the fluid velocities are assumed to be small. However, the fluid velocities discussed above reach a substantial fraction of the local speed of sound near the top boundary. Figures 8a and 8b show the rms (solid line) and maximum (dotted line) Mach numbers of the fluid velocity as functions of height from runs c23 and c34, respectively. The slight increases in Mach number near the upper and lower boundaries, seen in both runs, are due to large horizontal velocities near the boundaries. Mach numbers range as high as 0.8 in c34, and are at least a few percent at all depths in both runs. Force balance in the vertical direction, averaged both over time and horizontal position, therefore contains a significant contribution from the gradient of the dynamic pressure, ρu_z^2, as well as of the pressure P. Denoting both a horizontal and temporal average in the momentum equation by angle brackets, we have

$$F_m = -P_z - G_\rho \quad , \qquad (3.3)$$

where $F_m = \partial_z < \rho u_z^2 >$ is the gradient of the dynamic pressure, $P_z = \partial_z < P >$ is the pressure gradient, and $G_\rho = g < \rho >$ is the gravitational force per unit area. In order to assess the relative importance of the dynamic pressure term we plot $-P_z/G_\rho$ and $-(P_z+F_m)/G_\rho$ as functions of height for each of runs c23 and c34 in Figures 9a and 9b, respectively. If our simulations were in perfect statistical equilibrium and if our process of averaging our data over horizontal position and time were perfect as well, then our plots of $-(P_z+F_m)/G_\rho$ should be horizontal lines at the value of unity. We find that the sum $-(P_z+F_m)/G_\rho$ is within 0.01% of unity at all depths more than 8 computational cells away from the upper or lower boundaries in both runs, while $-G_\rho/P_z$ (where the dynamic pressure term is neglected) can vary by more than a percent in c23, and by as much as 7% in c34, where convective velocities are higher. From the curves in Figure 9, one might nevertheless think the effect of the dynamic pressure term to be quite small, but we have found it to be significant when dealing with the interpretation of other small quantities, such as the superadiabatic temperature gradient, which play important roles in mixing–length theories.

If we consider the dynamic pressure term F_m as known, the vertical profiles of density and pressure may be derived from the vertical profile of any one thermodynamic variable, such as temperature or entropy. The vertical profiles of entropy, for runs c23 and c34 are shown in Figures 10a and 10b. Solid lines show the mean value, error bars the *rms* fluctuations around the mean, and dotted lines the extrema at each height. The entropy is nearly constant in the lower two thirds of both runs, indicative of efficient convection and the layer of gas being well mixed. Entropy fluctuations are largest near the top boundary, and there is a systematic decrease in entropy with height in this region. Correspondingly, the vertical temperature gradients are nearly adiabatic in the lower halves of both runs, and there are superadiabatic temperature gradients in the upper halves. The extent to which the temperature gradient is superadiabatic at each height reflects how strongly the convection is driven given the imposed convective flux and the mass which is locally available to carry it. The slight subadiabatic temperature gradients in the lower portions of our convective layers are related to the bottom hard wall boundary conditions and will be discussed later.

Figures 11a and 11b show the mean temperature gradients for runs c21-c23 and c31-c34 as functions of depth. Over the bulk of the convective layer, away from both the top and the bottom walls, these temperature gradients are nearly independent of grid resolution, and hence of the effective viscosity of the gas. This is particularly evident if the two highest resolution simulations are compared in each series. As is most easily seen for the series of runs c31-c34, in which the thermal conductivity is very small and the bulk of the heat flux through the layer is carried by convection, the numerical viscosity has the greatest effect in the upper region. We will see that this is the region most important for comparisons of our results with mixing–length models. Since such comparisons are a major motivation for this study, we were driven to the very high grid resolution of the c34 simulation. From the results plotted in Figure 11b, however, we feel that the grid resolution of this run is sufficient for this purpose, and no further grid refinements are necessary in order to obtain useful results. In the top half of each model, temperature gradients are seen to be superadiabatic. There is a subadiabatic region, $0.07 < z < 0.24$, in each of the runs c21-c23 and similarly for $0.07 < z < 0.32$ in runs c31-c34. In each case the subadiabtic region is within one pressure scale height of the lower boundary and is caused by the impenetrable wall and thermal boundary layer at that edge. This thermal boundary layer generates very hot and extremely buoyant elements of gas, which continue to drive the flow even in the presence of a slight subadiabatic region such as is present in our models. For example, in run c34, the the increase in entropy going down across the lower thermal boundary layer is five times larger than the increase in entropy going up across the entire subadiabatic region. Further, the *rms* fluctuations in the entropy in the center of the subadiabatic region in run c34 is three times larger than the total increase in entropy across it. Hence, these subadiabatic regions only slightly decrease, and do not reverse, the buoyancy of typical positively buoyant elements of gas going across them. These subadiabatic regions are consistent with the systems being in convective equilibrium. Similar subadiabatic regions are seen in the lowest pressure scale height adjacent to an impenetrable boundary by other investigators both in 2-D (Hurlburt et. al. 1984) and in 3-D (Chan and Sofia 1989) simulations of convection.

In our discussion of mixing–length ideas, we will follow the simplified treatment given in Clayton 1968. More compressive developments of mixing–length theory can be found, for example, in Cox and Giuli 1968 or Hansen and Kawaler 1994, but we have found that the additional terms which these discussions introduce in the equations are generally negligible in the context of our simulations, presumably because in all the runs presented here we have chosen relatively small thermal conductivities, leading to highly efficient convection. Mixing–length theory is used to derive the entropy stratification in a convectively unstable zone of a star by relating the excess of the temperature gradient above its adiabatic value, $(\beta - \beta_{ad})$, to the enthalpy flux, F_E, carried by convection through the layer. The imposed total energy flux, F_T, consists of 3 contributions:

$$F_T = F_E + F_K + F_R \quad , \tag{3.4}$$

The total energy flux is independent of depth in our numerical simulations. However, the 3 terms on the right in equation (3.4) show interesting depth dependence which mixing–length theory attempts to capture by the relationships given below, with the coefficients α_E and α_K being constants in the theory. Here we consider these coefficients as functions of depth, and possibly of the ratio F_R/F_T. We then can measure the usefulness of mixing–length theory in describing the statistically averaged behavior of our simulations by the accuracy to which we can approximate these coefficients by constants. The 3 contributions to the total energy flux are the enthalpy flux

$$F_E \;=\; < c_p \rho \delta T u_z > \tag{3.5a}$$

$$F_E = \alpha_E c_p \;<\rho><\delta T^2>^{1/2}<\delta u_z^2>^{1/2} \tag{3.5b}$$

the kinetic energy flux

$$F_K \;=\; \frac{1}{2} <\rho u^2 u_z> = \frac{\alpha_K}{2} <\rho><\delta u_z^2>^{3/2} \tag{3.6}$$

and the radiative flux

$$F_R \;=\; -\kappa\, \partial_z \;<T> \tag{3.7}$$

Figures 11c and 11d show these vertical energy fluxes for runs c23 and c34. In the relations given above, these vertical energy fluxes, which have been averaged both over time and over horizontal position, are expressed in terms of products of averaged density, temperature, and velocity fluctuations. These averaged fluctuations must in turn be related to the deviation of the local temperature gradient from its adiabatic value if our mixing–length analysis is to be complete.

For the purpose of comparing the scalings predicted by mixing–length theory with data from our numerical models, we evaluate the temperature and velocity fluctuations in terms of the *rms* fluctuations of each about its horizontal mean, which we write as $< \delta T^2 >^{1/2}$ and $< \delta u_z^2 >^{1/2}$ in equations (3.5) and (3.6) above. Figure 12a shows values for α_E and α_K for each of the runs c23 ($F_R/F_T = 0.8$) and c34 ($F_R/F_T = 0.08$) as functions of the vertical coordinate z. We see that $\alpha_E \approx 0.8$, and $\alpha_K \approx 1.6$ for $z < 0.84$ For $z > 0.84$ α_K drops off rapidly. This is very likely due to the presence of the upper hard wall boundary. The tick marks along the bottom of Figure 12a show the depths at which the pressure P is 2, 4, 8, and 16 times that of the pressure P_T along the top boundary for both runs c23 and c34. In the absence of a hard wall boundary, the mean pressure at a given depth is proportional to the total mass above that depth. Therefore, as we go down into the unstable layer, and P increases beyond its value P_T at the top boundary due to the weight of the gas between us and that boundary, the difference between this bounded flow and one with a free surface should diminish. We may therefore adopt the ratio P_T/P as an estimate of the influence on the flow of the top hard wall boundary condition. When the inverse of this ratio, P/P_T, is large, a mixing–length analysis should be more likely to apply. Indeed, as we can see from Figure 12, when $P/P_T > 4$ the coefficients α_E and α_K become fairly independent of depth in run c34, and match up with the coefficients from run c23.

In comparing our simulation data to a mixing–length theory, we should not only account for the distortions of the data, from this point of view, near the top boundary but from the lower boundary as well. Our hard wall at the bottom of the simulation domain also makes mixing–length theory inapplicable in that region. We have used the ratio P_T/P to estimate the importance of the upper boundary condition on our results at a given depth. To estimate the importance of the lower hard wall boundary condition we instead measure the height above that boundary in terms of the local pressure scale height, λ_p, defined by

$$\lambda_p = -\frac{P}{\partial P/\partial z} \quad . \tag{3.8}$$

Since mixing–length theory suggests that the scale of typical convective eddies is comparable to λ_p, clearly the theory cannot apply when we are within this distance of a hard wall boundary. We have noted earlier that only in the upper portion of our simulations are significant superadiabatic temperature gradients observed. Mixing–length theory relates statistical features of the flow, such as the energy fluxes F_E and F_K and the temperature and velocity fluctuations on which they depend, to the superadiabatic temperature gradient $(\beta - \beta_{ad})$. This gradient is assumed to drive the convection, since it is proportional to the degree of instability of the horizontally and temporally averaged entropy stratification. Therefore, as we have already pointed out, a local mixing–length theory cannot apply in the lower portion of our simulations, where this superadiabatic temperature gradient is not found. This lower 30% to 40% of our simulation volume, which is not included in Figure 12, is also within a single pressure scale height of the hard bottom boundary. Therefore we should not be surprised that the convective motions persist there even in the presence of a stabilizing entropy stratification. The convective motions in this region are driven by the vigorous convection further up in the layer and by the thermal boundary layer along lower boundary. In fact, our visualizations of this convective flow, which can be viewed on the video accompanying this paper, clearly show descending plumes of turbulent cool gas extending over multiple pressure scale heights. The inertia of these rapidly descending plumes easily overcomes the slight stabilizing influence of the horizontally and temporally averaged entropy stratification. We are currently analyzing results of further convection simulations in which the convectively unstable layer has a stable layer of gas, not a hard wall, beneath it. It will be interesting to discover the vertical extent of applicability of mixing–length theories in those numerical experiments. That work will be presented in a forthcoming article.

We have seen that the simple parametrizations of the energy fluxes given in terms of the coefficients α_E and α_K defined above are indeed useful in describing our simulations, so long as we restrict our attention to the appropriate region of the simulation volume. We now ask whether in this restricted region the average temperature and velocity fluctuations can be equally well described by simple relations to the superadiabatic temperature gradient $(\beta - \beta_{ad})$. We might expect that the temperature fluctuations would depend on this quantity, which is related to the strength at which the convection is driven, as well as on the coefficient of heat conduction, which tends to reduce the amount of heat transport which results from a given level of vigor of the convective motions. We can estimate the potential importance of any such terms in the coefficient of heat conduction κ by comparing the convective time scale to the the thermal relaxation time scale. A convective time scale τ_c is given by

$$\tau_C = \frac{\lambda_p}{u_f} \quad . \tag{3.9a}$$

Here, we have taken λ_p to be a typical size, and u_f defined in equation (3.2) to be a typical velocity. The time scale for thermal relaxation is

$$\tau_r = \frac{\lambda_p^2}{\nu_T} \quad , \tag{3.9b}$$

where $\nu_T = \kappa/(\rho c_P)$ is the thermal diffusivity. The ratio of the thermal to the convective time scale gives us a Peclet number

$$Pe = \frac{\tau_r}{\tau_C} = \frac{\lambda_p u_f}{\nu_T} \quad . \tag{3.9c}$$

Values of Pe range from 8.3×10^4 near the bottom of run c34 to 10 near the top of run c23. We may therefore conclude that convection is efficient throughout all of our models, and that thermally diffusive terms may be neglected in a mixing–length treatment. We should point out, however, that the above analysis does not account for the strong turbulence that we observe in our convectively unstable atmospheres. This turbulence acts to enhance the local diffusion of heat above the level indicated by the coefficient of heat conduction. This effect reduces the efficiency of the convection significantly from the estimate given above, although it is possible to consider this effect as a fundamental aspect of the convection itself, at least in our regime of extremely low viscosity and hence of highly turbulent convection.

In the absence of any diffusive terms, the factors which should control the amplitude of temperature and velocity fluctuations are the local pressure scale height λ_p and the deviation of the temperature gradient β from its adiabatic value β_{ad}. From simple dynamical arguments, which amount to little more than unit analysis, we can write

$$< \delta T^2 >^{1/2} = \alpha_T (\beta - \beta_{ad}) \lambda_p \qquad (3.10)$$

for the *rms* temperature fluctuations, and

$$< \delta u_z^2 >^{1/2} = \frac{\alpha_u}{2} \left(\frac{g}{T} (\beta - \beta_{ad}) \right)^{1/2} \lambda_p \quad (3.11)$$

for the *rms* velocity fluctuations. Given hydrostatic equilibrium, the adiabatic temperature gradient is $\beta_{ad} = \frac{g}{c_v \gamma}$. However, as we have discussed earlier, the presence of velocity fluctuations leads to a dynamic pressure term in the momentum flux which modifies the adiabatic vertical temperature profile, and leads to

$$\beta_{ad} = \frac{g}{c_v \gamma} f_k \qquad (3.12)$$

where

$$f_K = \left(1 + \frac{1}{g\rho} \frac{\partial}{\partial z} (\rho u_z^2) \right) \quad . \qquad (3.13)$$

The additional factor, f_K, in equation (3.13) makes our analysis slightly nonlocal, but we have seen earlier that this is a small, yet nevertheless important, effect.

Figure 12b shows the factors α_T and α_u for runs c23 and c34 as functions of depth. We measure the average values of α_T and α_u over $0.68 < z < 0.84$ for run c23 and over $0.55 < z < 0.84$ for run c34. The upper bound of $z = 0.84$ for both runs is chosen for the same reasons as mentioned earlier. The lower bounds reflect the range of z over which there is a significantly superadiabatic temperature gradient. These ranges of z and the corresponding average values of α_T and α_u are shown in Figure 12b. From these averages we get $\alpha_T = 2.04$ and $\alpha_u = 2.70$

The relations for vertical energy fluxes given in equations (3.4) through (3.7) and the relations for velocity and temperature fluctuations given in equations (3.10) and (3.11) may be combined to relate the total energy flux to the temperature gradient in a form very similar to a local mixing–length model for efficient convection (see, for example, Clayton 1968 or Cox and Giuli 1968)

$$F_T = \alpha_{eutk}\ c_p \rho \left(\frac{g}{T} \right)^{1/2} (\beta - \beta_{ad})^{3/2} \lambda_p^2 \ + \ \kappa\beta \quad . \qquad (3.14a)$$

$$\alpha_{eutk} = \left[\frac{\alpha_E \alpha_U \alpha_T}{2} - \alpha_K \alpha_U^3 \frac{\gamma - 1}{16\gamma f_K} \right] \quad (3.14b)$$

Here, we have combined the kinetic energy flux with the enthalpy flux by using the definition for the pressure scale height λ_p given in equation (3.8) and including the dynamic pressure into the relationship between the pressure gradient and the density

$$\frac{\partial p}{\partial z} = g\rho \left(1 + \frac{1}{g\rho} \frac{\partial}{\partial z} (\rho u_z^2) \right) \quad . \qquad (3.15)$$

Given only an imposed total energy flux, which is independent of depth, and the mean density and pressure at one height (to establish units and vertical location) we can derive a vertical profile for the temperature gradient by using equations (3.11), (3.14), (3.15) and the equation of state. We calculate self-consistent solutions to these equations iteratively for models c23 and c34. Iterative solutions work here because the influence of the dynamic pressure is small in the sense that $f_K \approx 1$. We use an adiabatic atmosphere with zero velocity fluctuations for the initial iteration. Solutions converge within a few iterations. These mixing–length model solutions are compared with the 3-D numerical models of c23 and c34 (see table 1) in Figure 12c. We find that this local mixing–length model works fairly well over a range of depths in each of two numerical models with very different convective fluxes. The dotted line in Figure 12c is β_{ad} in the absence of dynamic pressure adjustments, while the two dashed lines are β_{ad} for each of the two numerical simulations (as indicated) including the effects of dynamic pressure.

We note that the main effect of dynamic pressure in equation (3.14) is in the $(\beta - \beta_{ad})$ term, where the deviations of β and β_{ad} from the value of 0.6 are both small. The enthalpy flux and kinetic energy flux, however, are not close to cancelling, hence the coefficient on the right hand side of equation (3.14) is essentially uneffected by the dynamic pressure term, and becomes

$$\left[\frac{\alpha_E \alpha_U \alpha_T}{2} - \alpha_K \alpha_U^3 \frac{\gamma - 1}{16\gamma}\right] = 1.28 \quad . \quad (3.16)$$

Using the MLT proportionality constants of Cox and Giuli 1968 in the case of large convective efficiency the convective flux can be written as

$$F_C = F_E + F_K = \frac{a_0 Q^{1/2}}{9\sqrt{2}} \frac{g^2 \rho^{5/2} T}{p^{3/2}} (\nabla - \nabla_{ad})^{3/2} \, \alpha\lambda_p \ , \quad (3.17)$$

where the coefficient of thermal expansion, Q, is unity for our polytropic gas, and $\Lambda = \alpha\lambda_p$ is the effective "mixing–length" used in Cox and Giuli 1968. With the choice of $a_0 = 9/4$ used in Cox and Giuli 1968, equation (3.17) is equivalent to equations (3.14) and (3.16) given a ratio of mixing–length to pressure scale height of $\alpha = 2.68$. The effective mixing–length, consistent with the numerical models of compressible convection presented here and the coefficients used in Cox and Giuli 1968 scales with, and is several times larger than, the local pressure scale height.

3.3. Convection and Turbulence

Vorticity is a useful diagnostic for identifying turbulent regions within flows which are both three dimensional and at a high Reynolds number. Vortex stretching insures that the largest local rates of strain and Reynolds stresses are associated with vortex tubes, which therefore play an important role in the behavior of turbulent flows. As we have seen in section 3.1, our models of thermally driven compressible convection in deep atmospheres generate narrow and turbulent downflow lanes. Vorticity is strong in these downflow lanes. We would like to understand how the vorticity comes to be concentrated in these regions. This vorticity could be generated by the strain fields associated with the downflow lanes, or it could simply be advected from the upper boundary. Close to the upper boundary, the baroclinic terms of the governing dynamical equations generate vorticity around drips of cold gas which form in the convectively unstable thermal boundary layer there. Compressional terms related to the mean density stratification also influence to the evolution of enstrophy. In this section we characterize the contribution of each of these mechanisms. Vorticity is a vector quantity and the geometry of a gravitationally stratified convective flow is far from isotropic. There are preferred directions of vortex stretching in the strongly anisotropic strain fields, which leads to to alignment of the vorticity with large-scale convective structures, as we show below.

Figures 13a-13h show eight perspective volume rendered visualizations of the vorticity field from run c34 at $t = 44$. Panels 13a through 13d show four horizontal sections, each section spans the entire horizontal extent of the model. Panel 13a shows the top pressure scale height which spans $z \in [0.87, 1.0]$, panel 13b shows the second pressure scale height from the top which spans $z \in [0.68, 0.87]$, panel 13c shows the third pressure scale height from the top which spans $z \in [0.39, 0.68]$, and panel 13d shows the lowest pressure scale height which spans $z \in [0.0, 0.39]$. The top pressure scale height of the model, Figure 13a, shows a network of vortex tubes. As seen in Figures 14a and 14d, below, these vortex tubes are associated with small downflow lanes in the thermal boundary layer. Where these small downflow lanes intersect, a relatively strong downflow plume results, which penetrates into lower layers of gas and produces turbulent vortex structures as seen in Figure 13b. The major downflow lanes are most clearly evident in Figure 13c where they are seen as lanes of tangled vortex tubes. Only the strongest portions of the downflow lanes penetrate to the deepest scale height shown in Figure 13d. The intersection of the major downflow lanes, seen at the upper left in Figure 2a, forms the strongest downflow plume which penetrates all the way to the bottom of the convective layer. Panels 13e through 13h show four vertical sections for ranges of the horizontal coordinate y in the intervals $[0.0, 0.25]$, $[0.5, 0.75]$, $[1.0, 1.25]$, and $[1.5, 1.75]$. The bundles of vorticity are primarily associated with downflow lanes. These vertical sections show vertically oriented vortex tubes extending up from the lower boundary at the center of the strongest upflow region. Vorticity tends to be aligned with the direction of large scale expansion. At the base of the upflow region the flow is primarily expanding in the vertical, and along the top boundary the flow is expanding in the horizontal, except in the downflow lanes. Both the vertical and horizontal sections show only weak vorticity in the center of the upflow region.

In the accompanying video we present a movie of the time dependent vorticity field of run c34 during the time interval $0 < t < 33.5$, see appendix A. The movie shows perspective volume renderings of the magnitude of vorticity, which visualizes vortex tubes and slip surfaces. After showing the full volume of the simulation, the movie zooms in to show detail in a vertical section of the simulation's rectangular volume. The full volume of the simulation is four times thicker than the subsection shown in this sequence. Note the numerous vortex tubes in the upper boundary. Near the center of this vertical section a cascade of vortex tubes is seen tumbling down, this is a cross–section of a turbulent downflow lane. The mostly clear regions on either side are where the gas is welling up.

Next, the movie moves to show vorticity in the upper $1/8^{th}$ of the simulation. Many long, horizontally oriented, and very strong vortex tubes can be seen here. These vortex tubes start as slip surfaces along the edges of downflow lanes of cool gas. Many small downflow lanes form along the upper boundary. But the overall flow which is associated with the largest convection cell stretches these downflow lanes into long structures. As they stretch, these slip surfaces tend to compress in the two directions which are orthogonal to the stretching, and, due to conservation of angular momentum, spin up into strong vortex tubes. The vortex tubes seen here are strong enough to form recirculation regions, which are essentially small convection cells embedded within a much larger convection cell. The concentrated bundles of vortex tubes, toward which the expanding patterns of vortex tubes flow, delineate the main downflow lanes of the single large convection cell which spans the depth of the layer. The bright dots, seen in the views of vorticity from above, are the ends of very strong vertically oriented vortex tubes which terminate at the upper boundary. The interplay and merger of these vertical gyres is clearly seen later in the movie where the view moves to examine vorticity in the upper boundary seen at an inclined angle.

As mentioned above, and as seen the movie of vorticity, there is a positive correlation between downflows and enstrophy (the square of the vorticity). We draw a more direct comparison of enstrophy and vertical velocity in Figures 14a-14f. Figures 14a and 14b show enstrophy in a horizontal section at mid depth and a vertical section, while Figures 14c and 14d show u_z in the same horizontal and vertical sections. Two intersecting downflow lanes are clearly seen in 14c. A corresponding increased density and strength of vortex tubes is seen in 14a. Similarly, a section of a downflow lane is seen (darker shades of gray) in 14d, with the corresponding increase in enstrophy in 14b. The downflow lanes are sufficiently turbulent that at a given instant in time there can be positive vertical velocity embedded within the downflows, which is seen Figures 14c and 14d as lighter shades of gray within the large scale downflow lanes. These small regions of counterflow are often due to very strong vortex tubes dominating the local flow, and do not correspond to any long lived updrafts. We can get a much cleaner, and more representative, image of the large scale circulation by filtering the data. Filtered vertical velocity, in the same horizontal and vertical sections, is shown in Figures 14e and 14f. For these models of compressible convection we use a Favre, or mass weighted, filter. Given any field quantity, Q, and any filter which produces, $\overline{Q}$, the Favre filter of Q is defined by

$$\tilde{Q} = \frac{\overline{\rho Q}}{\bar{\rho}} \quad . \tag{3.18}$$

We choose a Gaussian filter. Since there are systematic trends in the vertical direction, we only filter in the two horizontal directions. Hence, the filter we use is

$$\overline{Q}(x, y, z) = \tag{1}$$

$$N \int\int e^{-((x_1-x)^2+(y_1-y)^2)/\delta_k^2} Q(x_1, y_1, z) dx_1 dy_1, \tag{3.19}$$

where $\delta_k = 2\pi k/L$, and L is the periodic horizontal width of the simulation region. We use the Favre filtered velocity, $\tilde{u}_z$, filtered at $k = 4$ (as shown in Figures 14e and 14f) to distinguish between large scale downflows and upflows. As can be seen in Figure 14f the filtered velocity is smooth in the vertical direction, despite the fact that no filtering was done in the vertical.

We can quantify the correlation between downflows and enstrophy. In Figures 15a through 15e we show the mean enstrophy (solid curve) as a function of $\tilde{u}_z$ in five narrow bands of depth ranging from the top boundary shown in Figure 15a to the bottom boundary shown in Figure 15e. Downflows correspond to $\tilde{u}_z < 0$. The varying ranges of $\tilde{u}_z$ in the different depth intervals reflect the dependence of the strength of the flow with depth. Everywhere there is a strong trend of increasing enstrophy with decreasing $\tilde{u}_z$. Away from the vertical boundaries (Figures 15b, 15c, and 15d) the enstrophy contrast is a factor of 15 from the strongest downflows to the strongest upflows. This confirms the visual impression given by the volume renderings of vorticity in Figures 13a-13h and in the movies of vorticity on the companion video to this article.

The dotted curves in Figures 15a – 15e show the dependence on $\tilde{u}_z$ of the contribution, ω_z^2, to the enstrophy from the vertical component of the vorticity. We have scaled ω_z^2 by a factor of three for comparison with the total enstrophy. There is a systematic trend for this component of the enstrophy to be preferred in downflows in the upper regions ($3\omega_z^2/\omega^2 = 1.20$) and in upflows in the lower regions ($3\omega_z^2/\omega^2 = 1.29$). In visualizations of the vorticity, this preference shows up as strong vertical vortex tubes at the tops of downflows (Figure 13a) and at the bottoms of upflows (see the bottom right-hand sides of Figures 13e and 13g). Similarly, the horizontal components of the vorticity contribute preferentially to the enstrophy in upflows in the upper regions ($3\omega_z^2/\omega^2 = 0.40$) and in downflows in the lower regions ($3\omega_z^2/\omega^2 = 0.71$). Again, visualizations show this trend in terms of the visually striking horizontally oriented pairs of vortex tubes in the uppermost regions (Figure 13a).

We can understand the correlation of enstrophy with downflows by examing the terms in the dynamical equations which generate vorticity. By taking the curl of the momentum equation we can derive an equation for vorticity which looks like

$$\partial_t \omega + \mathbf{u}\cdot\nabla\omega = \nabla P \times \nabla \frac{1}{\rho} + \omega\cdot\nabla\mathbf{u} - (\nabla\cdot\mathbf{u})\omega \ . \tag{3.20}$$

The left hand side of this equation is the commoving time rate of change of the vector vorticity ω, the three terms on the right hand side of this equation are the baroclinic, vortex stretching, and divergence terms. The barioclinic term tells us where the convective instability generates vorticity, the vortex stretching term indicates where the strain field is amplifying vorticity, and the compressional term shows where vorticity is being concentrated or spread out. We can write a scalar equation for the local enstrophy, ω^2, by taking the dot product of equation (3.20) with ω, which produces

$$\frac{d\omega^2}{dt} = B + S + C \ , \tag{3.21}$$

where d/dt is the commoving derivative. Here, B is the baroclinic contribution

$$B = 2\omega \cdot \nabla P \times \nabla \frac{1}{\rho} \ , \tag{3.22}$$

S is the vortex stretching contribution

$$S = 2\omega \cdot \omega \cdot \nabla\mathbf{u} \ , \tag{3.23}$$

and C is the compressional contribution to the enstrophy

$$C = -2\omega^2 \nabla \cdot \mathbf{u} \quad . \tag{3.24}$$

Figures 16a-16e show the stretching, compressional, and baroclinic terms as functions of the filtered vertical velocity $\tilde{u}_z$ in the same five depth ranges as used in Figures 15a-15e. Semi–log axes are used here in order to display the large dynamic range in the source terms of enstrophy. Everywhere, at all depths and independent of vertical velocity, the stretching term S (solid curve) is the strongest of the three terms. The contribution of vortex stretching to the vorticity is proportional to the vorticity itself. Hence, as soon as any vorticity is produced, vortex stretching quickly amplifies it. Vortex stretching is the strongest in downflows, with the contrast in S from the strongest downflows to the strongest upflows being as large as a factor of 100. The baroclinic term B (dashed curve) is strongest near the top boundary where, as was seen in section 3.1, temperature gradients are the steepest and the convective instability is the strongest. The baroclinic term is also enhanced near the top boundary by the low densities there. There is also a trend for the baroclinic term to be stronger in downflows at all depths, except along the bottom boundary. The compressional term C (dotted curve) contributes to the enstrophy in downflows, where the flow is systematically compressing, and is negative (i.e., acts to decrease the enstrophy) in upflows, where there is systematic expansion of the fluid. In these semi–log plots we plot $log_{10}|C|$, and negative values of C are indicated with "x"s. Compressional effects systematically enhance the enstrophy in downflows, and diminish enstrophy in upflows. The scenario suggested by this data is consistent with what is seen in the movies of enstrophy. The baroclinic term generates enstrophy in the very smooth flow which reaches the upper boundary layer. Vortex stretching quickly amplifies the enstrophy, which is then advected into downflow lanes by the large–scale circulation. All three terms (S, C, and B) act to enhance the enstrophy in downflow lanes, only viscous dissipation of enstrophy, which scales as ω^2, keeps the enstrophy from exponentially increasing indefinitely. Along the lower boundary the large–scale circulation advects the strong enstrophy at the base of the downflows into the upflows. Viscous dissipation and expansion act together to diminish the enstrophy in upflows. Near the top boundary the effects of expansion are sufficiently strong to create an extremely smooth flow near the top boundary.

The preferential alignment of vorticity with the horizontal or vertical direction in identifiable regions of the flow, as seen in visualizations of vorticity and the trends of ω_z^2/ω^2 with $\tilde{u}_z$ (Figures 15a–15e) can be understood in terms of the strain fields associated with the large–scale circulation. As was seen in Figures 16a–16e, the vortex stretching term S dominates the production of enstrophy. The stretching term is strongest for vorticity aligned with the principal direction of strain. Figures 17a–17e show several measures of the rate of strain tensor $A_{ij} = \partial u_i / \partial x_j$ as functions of the filtered vertical velocity in the same five depth intervals as used in Figures 15 and 16. The solid curves represent the trace of A_{ij}, which is the divergence of velocity, and show the obvious trend of fluid expansion in upflows and compression in downflows. The dashed curve represents the vertical component of the divergence A_{33}, which is positive (expansion in the vertical) at the tops of the downflows and bottoms of the upflows, leading to maximal vortex stretching in the vertical direction in these two regions, and the alignment of vorticity seen in Figures 15a and 15e. The dotted curve shows the sum of the horizontal components of the divergence, which is positive (expansion in the horizontal) at the bottoms of the downflows and tops of the upflows (Figures 17a, 17b, and 17e), similarly leading to maximal stretching in the horizontal direction in these two regions, and vorticity preferentially horizontally aligned there.

While A_{ij} is the rate of strain of the unfiltered velocity, the values of A_{ij} plotted in Figures 17a through 17e are averages over sizable domains and reflect the large–scale strain field. In fact, the trends for A_{33} and $A_{11} + A_{22}$ at the tops and bottoms of up- and down-flows, discussed above, are just what one would get from the flow of the simplest convection cell that may be constructed with sinusoidal variations. The alignment of the vorticity with the principal direction of the large-scale strain, which we have pointed out in relation to Figures 15 and 17, reflects a general tendency which has also been observed in our simulations of homogeneous, compressible turbulence (Porter et. al. 1998). If unfiltered velocity data is used to construct the strain field, then strong vorticity in turbulent flows is preferentially aligned with the intermediate direction of the rate of strain tensor (Kerr 1987). This result is caused by the dominance of the local strain field by the circulating flow about a strong vortex tube, which sets up a principal direction of strain right near the vortex tube which is perpendicular to the direction along its length. By constructing the rate of strain tensor from a filtered velocity, one which is averaged over sizable domains as we have done here, one may remove the influence of the small-scale self–induced strain fields of individual strong vortex tubes. Using this approach in the context of a PPM simulation of homogeneous, compressible turbulence on a grid of 1024×1024×1024 cells, Porter et. al. 1998, showed that the local vorticity is indeed aligned with the principal direction of strain for the large–scale velocity field. This finding, along with that reported above for our compressible convection simulations confirms the expectation that vortex stretching is the dominant mechanism for amplifying vorticity in turbulent flows.

While thermodynamic fields are coupled to velocity through baroclinic (including buoyancy driving) and acoustic terms, effects of hysteresis, thermal diffusivity, and even shock heating allow thermal variations to be, in principle, independent of the velocity field. Hence, measures of thermal fluctuations help diagnose the structure of compressible convection. Indeed, the enthalpy flux, represented by the term F_E in equations (3.5), is just the correlation of the vertical velocity with thermal fluctuations. By examining the mean temperature variation as a function of the filtered vertical velocity, and also the temperature fluctuations about that mean for each filtered vertical velocity bin, we can identify where (relative to the large scale convection cells) and on what size scales (relative to the width of the filter) the enthalpy flux is occurring. Figures 18a – 18e show the trend of the relative temperature fluctuations with filtered vertical velocity in the same five depth ranges as used in Figures 15, 16, and 17. The solid line shows the average value of $\delta T / < T >$ in each velocity bin, where $< T >$ is the mean temperature at each depth and

δT is the mean fluctuation of the temperature from $< T >$ in each velocity bin. The error bars show the *rms* fluctuations about mean temperature (i.e., $< T > +\delta T$) within each velocity bin at each depth range. At all depths, temperature fluctuations are predominantly hot in upflows and cool in downflows, indicative of positive enthalpy flux in both downflows and upflows. However, despite the fact that the mean temperature in upflows is relatively hot at each depth, the temperature fluctuations about this relatively hot mean are small compared to those in the downflows : the small scale thermal turbulence is relatively weak in the upflows compared to the downflows. This trend is seen at all depths, except along the lower boundary, Figure 18e. Away from the top and bottom boundaries the local *rms* temperature fluctuations are typically 7 times larger in the strongest downflows than in the strongest upflows. While there is plenty of positive enthalpy flux in the upflows (due to the mean positive temperature fluctuations in the upflows, see solid lines in Figure 18), there is – evidently – very little small scale thermal structure in the upflows. Hence, we find that both velocity and thermal turbulence is strongest in the downflow lanes, while the updrafts are relatively smooth. Even in the strongest downflows the *rms* temperature fluctuations are smaller in amplitude than the average relatively cool temperature variation associated with those negative vertical velocity bins. Hence, enthalpy flux comes primarily from the largest scale of convective motions (i.e., the mean flow of the large convection cells which span several pressure scale heights) as opposed to many small and independent contributions from the smallest scales of turbulence.

4. Conclusion

We have presented several high resolution models of three–dimensional, compressible, and thermally driven convection in deep atmospheres. Spectra of the vertical component of the velocity in a 2-D slice at mid layer have been compared for two series of simulations covering a range of a factor of 8 in grid resolution. These show not only convergence of the results to a common spectrum as the grid is refined, but they also indicate a power-law behavior with the energy per mode in a 2-D slice scaling as $k^{-8/3}$. If isotropy is assumed, this is consistent with energy per mode scaling as $k^{-11/3}$ for the velocity field in the full 3-D volume. This power-law scaling of velocity is consistent with a Kolmogorov energy spectrum (i.e., $k^{-11/3}$ energy per mode or $k^{-5/3}dk$ energy in a spherical shell of thickness dk), and holds over more than a decade of size scales in the highest resolution model presented here. Kolmogorov-like energy spectra have also been noted by other investigators (Cattaneo et. al. 1991 and Chan and Sofia 1989). However, we find that about half of the buoyancy driving of the kinetic energy comes from the same range of scales in which the the velocity spectra are Kolmogorov-like. Further, the spectra of the horizontal components of velocity are less than the that of the vertical component by as much as a factor of two in this same range, indicating significant anisotropy. Hence, one should be careful in interpreting a Kolmogorov-like energy spectrum as the sole indicator of a Kolmogorov inertial range, especially in models of convection in deep atmospheres where there is strong gravitational stratification and broad band energy driving.

Our convection flows possess intensely turbulent downflow lanes and relatively laminar updrafts. Vorticity is everywhere preferentially aligned with the strain field associated with the large–scale circulation of the convection. Near the top boundary, where the strain field is particularly large, the alignment of vorticity with strain leads to very strong, horizontally oriented vortex tubes. These horizontal vortex tubes come in counter–rotating pairs and are associated with downflow lanes.

Comparisons of our simulations with mixing–length theory show that for systems where radiative transfer carries only 8% as well as for systems in which it carries 80% of the total energy flux, over a factor of 8 in mesh resolution, and at all depths sufficiently well separated from the hard wall boundaries, the mean temperature fluctuations and vertical velocity fluctuations can separately and fairly accurately be related to the superadiabatic temperature gradient ($\Delta\nabla$) and the local pressure scale height (λ_p). Additionally, these fluctuations are correlated with the enthalpy and kinetic energy fluxs, which allows one to relate the convective energy flux (F_C) to $\Delta\nabla$ and λ_p in a manner consistent with mixing–length theory. From equation (3.17), which uses proportionality constants given by Cox and Giuli 1968, we find a value of $\alpha = 2.68$ for the ratio of mixing–length to the local pressure scale height in the case of large convective efficiency.

By way of comparison with previous work, Chan and Sofia 1989 used 3-D numerical simulations on computational meshes no larger than $28 \times 28 \times 46$ and inferred a mixing–length ratio of $\alpha = 2.3$ for the same MLT proportionality constants. Chan and Sofia 1989 also generated best fit relations between many combinations of dynamical variables, some of which were similar to those reported here. Remarkably, they found a correlation coefficient of 0.81 between vertical velocity and temperature fluctuations, essentially equivalent to our value of $\alpha_E \approx 0.8$ in equation (3.5) if density fluctuations can be ignored. Later, using higher mesh resolution ($137 \times 137 \times 100$) with convectively stable vertical boundaries, Chan and Sofia 1996 re-examined thier relations between dynamical variables and found that small to moderate adjustments of these relations could be made, but cautioned against putting to much weight on this "fine tuning" because of ambiguities in the subgrid scale model that they used. Our highest resolution 3-D models (c34 in Table 1 for example) compliment the subgrid scale models used in Chan and Sofia 1989 and Chan and Sofia 1996 by directly simulating an extended range of turbulent scales of fluid motion on regular and cubical meshes. For the purpose of constructing models of stellar evolution, our simulations therefore provide both added confidence in the classic mixing–length approach and improved quantitative estimates for the parameters involved.

The imposed energy flux and low density near the top boundary of the models in which the convective energy flux is dominant (runs c31 to c34, see Table 1) lead to flows with peak Mach numbers approaching unity in this region. The resulting vertical dynamic pressure alters the horizontally and temporally averaged vertical profiles of density and pressure from those which would apply in a hydrostatic equilibrium. Although this effect is small, it is important in measuring the parameters of a mixing–length theory, which must describe other small quantities.

The models presented here all treat a single convectively unstable layer with impenetrable walls at the top and bottom, in terms of a simple γ-law gas with a constant coefficient of heat conduction. The lower impenetrable boundary causes there to be a slightly subadiabatic temperature gradient in the lowest pressure scale height (as is also seen by other investigators (Hurlburt et. al. 1984 and Chan and Sofia 1989), which limits the applicability of our models to the interior of stellar convection zones, well away from the base of the convection zone. We are currently analyzing models of convection where the lower boundary is replaced by a convectively stable layer and the local coefficient of heat conduction is derived from Kramers' opacity law. Our analysis of MLT in these paper is also constrained to modest values of the superadiabatic temperature gradient : the presence of an impenetrable wall at the top boundary limits our measurements to depths in which $\Delta\nabla \leq 0.01$ is satisfied. Larger values of $\Delta\nabla$ inevitably lead to large Mach numbers, which imply large fractional pressure variations. An accurate representation of the uppermost pressure scale heights near a stellar surface would require a vacuum, or better still photospheric, boundary which is free to move and flex in response to the turbulent convective motions deeper down. We are currently performing simulations of stellar convection in spheroidal geometry with freely moving photospheric boundaries.

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A. Movie

In this Appendix we describe development and features of the vorticity field from run c34 (see Table 1), a $512 \times 512 \times 256$ local area model of stellar convection. The accompanying video shows the evolution of the magnitude of vorticity $\omega = |\nabla \times \mathbf{u}|$, over the first 33.5 time units of the simulation in steps of 0.125 time units, where time units are based on the acceleration due to gravity and the depth of the layer. This time sequence is repeated a total of eleven times from varying viewpoints and showing various subregions of the computational domain. Perspective volume rendering is used to visualize the volumetric data of this 3-D numerical simulation. In terms of the equilibrium *rms* vorticity, ω_o, a region is rendered as transparent for $\omega < 1.68\omega_o$. Opacity linearly increases from 0 for $\omega > 1.68\omega_o$. Color ramps from blue through green to white as the magnitude of vorticity ramps from $1.68\omega_o$ through $3.05\omega_o$ to $\omega >= 3.88\omega_o$.

The initial state of run c34, which is seen at the beginning of each time sequence in the accompanying video, is the final and dynamically relaxed state of run c33 (see Table 1). The parameters of run c33 are identical to those of c34 except for mesh resolution, which in run c33 is 1/2 in each direction of that of run c34. Hence the fully developed vorticity field in run c33 is about 1/2 in amplitude of that in run c34. Correspondingly, relatively weak vorticity is seen at the beginning of each of the 11 time sequences in terms of a mostly transparent volume. The *rms* vorticity in run c34 increases from 55% to 95% of its' equilibrium value over the first four time units of the run.

Here, we shall choose coordinates so that the computational volume of run c34 is $(X, Y, Z) \in ([-0.5, 0.5], [-0.5, 0.5], [-0.25, 0.25])$. The initial viewpoint is at $(X, Y, Z) = (2.5, 2.0, 1.0)$, which is above the top boundary and looks diagonally down at the computational volume. The viewpoint moves, as the systems evolves, to $(X, Y, Z) = (0, 2, -1)$, which is below the bottom boundary and looking diagonally upward at a 30 degree angle. The movie then fades back to the initial state, but now as seen from this new point of view. The time sequence repeats with the viewpoint staying below the lower boundary and orbiting 90 degrees in the X-Y plane around the computational volume to $(X, Y, Z) = (-2, 0, -1)$. These first two time sequences show the entire flow in terms of strong vorticity. This thresholding reveils intense turbulence along the upper boundary as well as in the downflow lanes and plumes. Vorticity, strong or weak, is a fairly good tracer of the large scale flow in these movies since commoving vorticity tends to evolve on longer time scales than it takes to be carried a significant distance through the computational volume. Many downflow lanes emanate from the top boundary and merge, hierarchically, into fewer and larger downflows until they merge into a fairly intermittent downflow plume. Updrafts are seen to be relatively clear of strong vorticity except in the regions of outflow along the lower boundary from the bottom of the main downflow plume. Attempts to volume visualize the weak vorticity in the updrafts yield an opaque field where only the boundaries of the volume are seen, resulting effectively in visualizing vorticity on 2-D slices. See, for comparison, Figures 14a and 14b which show horizontal and vertical 2-D cuts of the magnitude of vorticity, both weak and strong, in terms of a linear grey scale ramp.

Next, the movie focuses on a section of a downflow lane by displaying only the subvolume $(X, Y, Z) \in ([0.125, 0.375], [-0.5, 0.5],[-0.25, 0.25])$, which is a vertical slice spanning one quarter of the X dimension of the computational volume. The viewpoint moves to (-1,0,0) to center this subvolume, and then the time sequence is run through three times. Again, intense vorticity is seen along the top boundary and in regions of downflow, while relatively weak vorticity is seen in updrafts.

Next the movie focuses on the turbulent upper boundary by displaying only the subvolume ([-0.5,0.5],[-0.5,0.5],[0.1875,0.25]), which spans the uppermost eighth of the computational volume. The viewpoint moves to (0, 0, 2) and looks straight down in order to center this horizontal slice. The time sequence is run through three times. Along the top boundary, the flow is seen to converge towards the main downflow lanes. Between the downflow lanes, where the flow is expanding in the two horizontal directions, pairs of vortex tubes can be seen to form and connect into a horizontal network.

Next the movie zooms into one quarter of the top boundary by displaying only the subvolume ([0,0.5],[-0.5,0],[0,0.25]), which is an octant of the entire volume. The viewpoint moves to (0.25, -0.25, 1.125) and looks straight down in order to center this octant. Then the time sequence is run through two more times. This zoom-in provides a better view of vortex tube pairs and the formation and evolution of vortex rings.

Next the movie shifts to view the entire top boundary again, showing only the subvolume ([-0.5,0.5],[-0.5,0.5],[0.1875,0.25]) as before, but this time from a point of view which is to one side at (-0.6, 0.1, 0.4187) looking diagonally downward. The time sequence is played through two more times from this viewpoint. Myriad vertical vortex tubes, which terminate on the free slip top boundary, are seen within this horizontal slice. These gyres are seen to converge toward the main downflow lanes. Strong stretching in the vertical at the tops of the downflow lanes aligns mainly vortex tubes in the vertical direction, and hence with each other. The interaction of these very strong and nearly vertical vortex tubes takes on a 2-D character. Pairs of co-rotating tubes orbit around each other and frequently merge. Occasionally, a counter-rotating pair is seen self-propagating over a short distance.

Finally, the vorticity field in the entire volume is shown at a fixed time while the view point orbits back, through about 270 degrees in the X-Y plane, to the initial viewpoint at (2.5,2.0,1.0).

Perspective volume rendered movies, like the one described here, have proven to be useful in motivating questions about, and developing diagnostics for, these intensely turbulent 3-D flows. In particular, the correlation of vorticity with downflows and the alignment of vorticity with the principal direction of strain of the mean flow were both first noticed in volume rendered movies of the magnitude of vorticity, such as the one shown here. The quantitative analyses in section 3.3 were motivated be these visualizations.

REFERENCES

Bogdan, T. J., Cattaneo, F., Malagoli, A. 1993, Astrophys. J. 407, p. 316

Bohm–Vitense, E. 1958, Zs. Ap., 46, p. 108

Borue V., and Orszag S. 1994, preprint, Princeton University

Brachet,M. E., Meneguzzi, M., Vincent, A., Politano, H., and Sulem, P.-L. 1992, Phys. Fluids A 4, p. 2845

Brummell, N. H., Xie, X., Toomre, J., Baillie, C. 1995, Astron. Soc. Pac. Conf. Ser. 76, p. 192

Canuto, V.M., Cheng, Y., Hartke, G.J., and Schilling, O. 1991, Phys. Fluids A, 3, pp. 1633

V.M. Canuto, and I. Massitelli 1991, Ap.J., vol. 370, pp. 295-311

V.M. Canuto, and I. Mazzitelli 1992, Ap.J., vol. 389, pp. 724-730

V.M. Canuto, I. Goldman, and I. Mazzitelli 1996, Ap.J., 473, pp. 550-559

Cattaneo, F., Brummell, N.H., Toomre, J., Malagoli, A., Hurlburt, N. E., 1991 Ap. J. 370, pp. 282-294

Cattaneo, F. 1992, Ap. J. 393, p. 165

Chan, K. L., and Sofia, S. 1986, Ap. J. 307 pp. 222-241

Chan, K. L., and Sofia, S. 1989, Ap. J. 336 pp. 1022-1040

Chan, K. L., and Sofia, S. 1996, Ap. J. 466 pp. 372-383

Clayton, D. D. 1968, *Principles of Stellar Evolution and Nucleosynthesis*, (The University of Chicago Press)

Cox, J. P., and Giuli, R. T. 1968, *Principles of Stellar Structure*, (New York: Gordon & Breach)

Colella, P., and Woodward, P. R., 1984, J. Comput. Phys. 54, p. 174.

Gagne Y. 1987, These d'Etat, Institut Polytechnique Universitaire de Grenoble, unpublished.

Glatzmaier, G. A., and Toomre, J. 1995, Astron. Soc. Pac. Conf. Ser. 76, p. 200

Hansen, C. J.,, and S. D. Kawaler, S. D. 1994, "Stellar Interiors: Physical Principles, Structure, and Evolution," Springer–Verlag

Hartke, G. J., Canuto, V. M., and Dannevik, W. P. 1988, Phys. Fluids, 31, pp. 256

Hurlburt, N. E., Toomre, J., and Massaguer, J. M. 1984, AJ282, no. 2, pt. 1, pp. 557-573

Hurlburt, N. E., Toomre, J., and Massaguer, J. M., and Zahn, J.-P. 1994, AJ421, pp. 245-260

Jimenez, J., Wray, A., Saffman P., and Rogallo R. 1993, J. Fluid Mech. 255, p. 65

Kerr, R. M. 1987, Phys. Rev. Lett., 59, p. 783

Kim, Y.-C., Fox, P., Demarque, P., and Sofia, S. 1996, Ap. J. 461 pp. 499-506

Kraichnan, R.H. 1964, Phys. Fluids, 7, p. 1030

Malagoli, A., Cattaneo, F., Brummell, N. H. 1990, Ap. J. 361, L33

Nordlund, A., 1985, Sol. Phys. 100, no.1-2, pp.209-235

Nordlund, A., and Stein, R. F. 1993, A & A 267, p. 265

Nordlund, A., Brandenburg, A., Jennings, R. L., Rieutord, M., Ruokolainen, J., Stein, R. F., and Touminen, I 1992, Ap. J. 392 p. 647

Orzag, S.A. 1977, "Statistical Theory of Turbulence", in Fluid Dynamics, Les Houches 1973, 237-374, eds. R. Balian and J.L. Peube. (New York: Gordon and Breach).

Porter, D. H., and Woodward, P. R. 1988, in High Speed Computing: Scientific Applications and Algorithmic Design (University of Illinois Press, Urbana and Chicago), p. 77

Porter, D.H., and Woodward, P.R 1989, "Simulations of Compressible Convection with PPM," in ACM SIGGRAPH Video Review Special Issue #44, "Volume Visualization State of the Art", ed. Laurin Herr 1989, Pacific Interface/Dupont.

Porter, D.H., Woodward, P.R, Yang, W., and Mei, Q. 1990, "Simulation and Visualization of Compressible Convection in Two and Three Dimensions," in Nonlinear Astrophysical Fluid Dynamics, Volume 617 of the Annals of the New York Academy of Science, December 1990, ed. R. Buchler, also AIAA paper No. 90-0209.

Porter, D. H., Woodward, P. R., and Mei, Q. 1991, Video J. Eng. Res. 1, No. 1, pp. 1-24.

Porter, D. H., Pouquet, A.,, and Woodward, P. R. 1992a, J. Theor. Comp. Fluid Dyn. Vol. 4, No. 1, pp. 13-49

Porter, D. H., Pouquet, A, and Woodward, P. R. 1992b, Phys. Rev. Lett. 68, 3156

Porter, D. H., Pouquet, A.,, and Woodward, P. R. 1994, Physics of Fluids A, 6, No. 6, pp. 2133-2142

Porter, D. H., and Woodward, P. R. 1994, Ap. J. Suppl. 93, pp. 309-349

Porter, D. H., Pouquet, A., and Woodward, P. R. 1995, "Compressible Flows and Vortex Stretching," in Proc. *Small-scale Structures in Three-Dimensional Hydrodynamic and MHD Turbulence*, Eds. M. Meneguzzi, A. Pouquet and P.L. Sulem, Springer-Verlag.

Porter, D. H., and Woodward, P. R., 1996: "Compressible Fluid Turbulence, Convection, and Vortical Structures," to appear in *New Tools in Turbulence Modelling*, Eds. J. Ferziger and O. Métais, Springer–Verlag.

Porter, D. H., Woodward, P. R., and Pouquet, A. 1998, "Inertial Range Structures in Compressible Turbulent Flows," Physics of Fluids, Vol. 10, Issue 1, pp. 237-245.

She, Z. S., and Jackson E., 1993, Phys. Fluids A, 5, p. 1526

Singh, H. P., and Chan, K. L 1993, A & A 279 pp. 107-118

Singh, H. P., Roxburgh, I. W., and Chan, K. L. 1994, A & A 281 pp. L73-L76

Singh, H. P., Roxburgh, I. W., and Chan, K. L. 1995, A & A 295 pp. 703-709

Sofia, S., and Chan, K. L., 1984, Ap. J. 282 pp. 550-556

Vitense, E 1953, Zs. Ap., 32, p. 135

Woodward, P. R., and Colella, P., 1984, J. Comput. Phys., 54, 115.

Woodward, P. R. 1986, in Astrophysical Radiation Hydrodynamics, ed. K.-H. Winkler & M. L. Norman (Dordrecht: Reidel), 245

Woodward, P. R., Porter, D. H., Ondrechen M., Pedelty J., Winkler, K-H, Zabusky, N. 1987, pp. 557-643 in Proc. Third International Symposium on Science and Engineering on Cray Supercomputers, Mn., September 1987, Published by Cray Research Inc., 608 Second Av. South, Minneapolis, MN, 55402, Ed. John E. Aldag.

Woodward, P. R., 1988: "Supercomputer Simulations in Astrophysics," a series of 3 hour-long video programs produced by and available from the University of Minnesota Department of Independent Study.

Woodward, P. R.,, Porter, D. H., Edgar, B. K., Anderson, S., Bassett, G. 1995, Comp. Appl. Math., 14, no. 1, pp 97-105.

Xiong, D.-R., 1989, A & A 213, no.1-2, pp. 176-182

Fig. A1.— Horizontal sections of vertical velocity (panels a, b, and c) and relative temperature (panels d, e, and f) from a simulation of compressible convection on a 256x256x128 mesh with the PPM code (run c23, $M_o = 0.021$, $Ra = 1.010 \times 10^{12}$, Pr$= 3.623 \times 10^{-5}$). Panels a, b, and c are at horizontal levels z=0.25, 0.5, and 0.934, respectively. Similarly, panels d, e, and f are at horizontal levels z=0.25, 0.5, and 0.934, respectively. Dark shades of gray represent downflows or relatively cool temperature, light shades of gray represent upflows or relatively hot temperature. Note the cellular pattern of downflows near the top boundary (panels 1a and 1d), the large-scale cellular pattern of downflows at mid depth which spans the horizontal extent of the model (panels 1b and 1d), and the strong correlation between downflows and relatively cool temperatures at all depths. The low effective Prandtl number in this model leads to higher levels of turbulence in the velocity field (see especially panels 1b and 1c) than in the thermal field (panels 1e and 1f).

Fig. A2.— Horizontal sections of vertical velocity (panels a, b, and c) and relative temperature (panels d, e, and f) from a simulation of compressible convection on a 512x512x256 mesh with the PPM code (run c34, $M_o = 0.035$, Ra$= 2.229 \times 10^{15}$, Pr$= 7.549 \times 10^{-5}$). Panels a, b, and c are at horizontal levels z=0.25, 0.5, and 0.975, respectively. Similarly, panels d, e, and f are at horizontal levels z=0.25, 0.5, and 0.975, respectively. Dark shades of gray represent downflows or relatively cool temperature, light shades of gray represent upflows or relatively warm temperature. Note the fine grained cellular pattern of downflows near the top boundary (panels 2a and 2d), the large-scale cellular pattern of downflows at mid depth which spans the horizontal extent of the model (panels 2b and 2d), and the strong correlation between downflows and relatively cool temperatures at all depths. The low effective Prandtl number in this model leads to higher levels of turbulence in the velocity field (see especially panels 2b and 2c) than in the thermal field (panels 2e and 2f).

Fig. A3.— Comparison of the orientation of downflow lanes between a model where the principal horizontal periodic directions are aligned with the mesh directions (panels a, b, and c) to a model where the principal horizontal periodic directions are at a 45^o angle to the mesh directions (panels d, e and f). Vertical velocity is shown in horizontal sections at depths z=0.5 (panels 3a and 3d), z=0.6875 (panels 3b and 3e), and z=0.875 (panels 3c and 3f). The two models are identical (runs c11 and c21 with $M_o = 0.035$, Ra$= 2.229 \times 10^{15}$, Pr$= 3.623 \times 10^{-5}$) except for horizontal mesh and boundaries. In both cases, and at all depths, downflow lanes tend to align with the principal horizontal periodic directions rather than with the mesh directions. Hence, the predominant alightment of the large-scale downflows in all of our models (c21 through c34, see table 1) is due to our modest choice of aspect ratio (i.e., $2 \times 2 \times 1$) for the volume of the simulation.

Fig. A4.— Vertical velocity spectra (mean square amplitude per Fourier mode) in a horizontal cut, z=0.5, from simulations of compressible convection (a) for $M_o = 0.21$ for three mesh resolutions (runs c21, c22, and c23, see table 1), (b) for $M_o = 0.35$ for four mesh resolutions (runs c31 through c34 see table 1), and the same sets of vertical velocity spectra as (a) and (b) but shifted to align dissipation ranges and compensated for the

$k^{-8/3}$ trend in panels (c) and (d). A $k^{-8/3}$ power–law is shown for comparison in panels (a) and (b).

Fig. A5.— Spectra of buoyancy forcing in two high resolution simulations of compressible convection at heights approaching the upper boundary. Power spectra of the mass density are shown at the three heights indicated in run c23 (panel a) and four heights indicated in run c34 (panel b). Significant buoyancy forcing is shown to span the entire range of scales over which the velocity spectra in Figure 4 can be said to have converged and possess a power–law form. Velocity power spectra in two high resolution simulations of compressible convection at six heights approaching the upper boundary as indicated in run c23 (panel c) and run c34 (panel d). Large-scale velocity modes are systematically suppressed down to smaller and smaller length scales as the upper boundary is approached, consistent with the constraint that the bulk of the velocity energy is in roughly circular eddies or vortex tubes.

Fig. A6.— Time histories of the total energy, panels (a) and (b), the mean temperature along the top, panels (c) and (d), and the total vertical energy flux at top panels (e) and (f), from simulations of compressible convection with $M_o = 0.21$, panels (a), (c), and (e), and with $M_o = 0.35$, panels (b), (d), and (f). Only the highest two resolution models are shown for each value of M_o (i.e., runs c22 and c23 for $M_o = 0.21$ and run c33 and c34 for $M_o = 0.35$). The times of the lower resolution runs are shifted so that the time at which they are used to generate the initial state of the next higher resolution run are at t=0. Hence the vertical dotted lines in each Figure represent mesh refinement by a factor of 2.

Fig. A7.— Vertical velocity, u_z, as function of height, z in two high resolution models of compressible convection, run c23 in panel a, and run c34 in panel b (see table 1). The mean, *rms*, and extremal values at each height are shown as a solid line, error bars, and dotted lines, respectively. A dashed line shows the vertical velocity, u_f in equation (3.2), derived from the demand that the entire convective flux, $(F_E + F_K)$, be carried solely by the kinetic flux, F_K. This simple form of u_f provides an overall scaling for the amplitude of fluctuations in the vertical velocity .

Fig. A8.— Local Mach number as function of height, z, in two high resolution models of compressible convection, run c23 in panel a, and run c34 in panel b (see table 1). *rms*, and maximum values at each height are shown as solid and dotted lines, respectively. The extreme Mach numbers approach unity near the upper boundary, where the speed of sound is the lowest and the mass density, which must carry the convective flux, is also the lowest.

Fig. A9.— Effects of convective momentum transport on the mean atmospheric profile in two high resolution models of compressible convection. Panel a shows results for run c23, while panel b shows results from run c34 (see table 1). Dotted lines show the pressure gradient scaled by the product $g\rho$. This ratio would be unity for a hydrostatic atmosphere, but is seen to vary by a few percent from unity, especially near the upper boundary and particularly in run c34, where Mach numbers are larger. Solid lines show the pressure gradient with the dynamic pressure gradient term added in, again scaled by the product $g\rho$. As it should be for long time-based averages over a turbulent, but statistically steady flow, this ratio is unity to within a fraction of a percent.

Fig. A10.— Vertical profile of entropy in two high resolution models of compressible convection, run c23 in panel a, and run c34 in panel b (see table 1). The mean, *rms*, and extremal values at each height are shown as a solid line, error bars, and dotted lines, respectively. Each atmosphere is seen to be nearly adiabatic, with small fluctuations, except in the uppermost pressure scale heights, where fluctuations are larger and the mean entropy decreases as the upper boundary is approached. The decrease in entropy near the upper boundary is larger in run c34 than c23, consistent with the atmosphere carrying a larger convective flux in c34.

Fig. A11.— Vertical profiles of the mean temperature gradients from simulations of compressible convection (a) for $M_o = 0.21$ for three mesh resolutions (runs c21, c22, and c23, see table 1), (b) for $M_o = 0.35$ for four mesh resolutions (runs c31 through c34 see table 1). The slope $dT/dz =$ -0.6 (solid straight line) corresponds to a static adiabatic atmosphere. The total F_T, radiative F_R, enthalpy F_E, and kinetic F_K vertical energy fluxes are shown for the high resolution simulations c23 (c) and c34 (d).

Fig. A12.— Correlation coefficients α_K and α_E (see equations (3.5) and (3.6)) for the two high resolution simulations c23 and c34 (see table 1) are shown in (a). The coefficients α_T and α_u (see equations (3.10) and (3.11)) for the same two simulations are shown in (b). Solid horizontal lines show the best fit values (and the corresponding ranges of z) of α_T and α_u for each of the two simulations. Local mixing–length model solutions (solid lines) for the adverse temperature gradient, β, (see equation (3.14)) are compared in (c) with measured β values ("x"s) from the two 3-D numerical models of c23 and c34 (see table 1). The dotted line is β_{ad} derived in the absence of dynamic pressure corrections, while the two dashed lines show β_{ad} for each of the two numerical simulations (as indicated) including the effects of dynamic pressure.

Fig. A13.— Perspective volume rendered visualizations of enstrophy (vorticity squared) in the high resolution model of compressible convection c34 (see table 1). Panels a, b, c, and d show horizontal sections which span the topmost, 2nd from the top, 3rd from the top, and bottom pressure scale heights, respectively. Panels e, f, g, and h show four vertical slices, each of which spans the computational volume in the vertical direction and one horizontal direction, and spans $1/8^{th}$ of the computational volume in the other horizontal direction. Note the cellular pattern of downflow lanes is seen in the vorticity field in the top three pressure scale heights (panels a, b, and c). There is a turbulent layer in the top few pressure scale heights, which is driven by the superadiabatic temperature gradient there (see Figures 10 and 11). Turbulent downflow lanes, surrounded by relatively laminar updrafts are seen in the vertical slices (see panels e, f, g, and h).

Fig. A14.— A visual comparison of the correlation between turbulence and downdrafts in the high resolution model of compressible convection c34 (see table 1). Turbulence is visualized in terms of strong enstrophy (dark shades of gray) in a horizontal section at mid layer (a), and in a vertical section (b), while downflow is visualized in terms of vertical velocity being very negative (dark shades of gray) in the same horizontal section at mid layer (c), and vertical section (d). The positive correlation between downflow and turbulence is fairly clear, however there are updrafts seen in the middle of downflow regions, due to the strength of the turbulence itself, that do not correspond to the larger mean flow. Panels (e) and (f) show filtered vertical velocity in the same horizontal and vertical sections as above. the filter is a Favre (or mass weighted) filter based on a Gaussian filter with a dispersion wavenumber $k = 4$ in the two horizontal directions only (see eqns (3.18) and (3.19)). The filtered velocity is quite smooth, even in the vertical direction where no filtering is done. There is still a clear visual correlation between enstrophy (panels 14a and 14b) and filtered vertical velocity.

Fig. A15.— Dependence of enstrophy on filtered vertical velocity in five horizontal slices. The filtered vertical velocity is the same as shown in Figures 14e and 14f, and corresponds to the overall large scale flow. Each slice is 0.0625 deep (or $1/16^{th}$ the depth of the layer). Panels a, b, c, d, and e show results for slices centered at $z = 0.953$ (near the top boundary), 0.875, 0.68, 0.391, and 0.039 (near the bottom boundary), respectively. Solid curves show the mean enstrophy, while dotted curves show three times the mean vertical component of enstrophy. At all depths there is a positive quantitative correlation between enstrophy and filtered vertical velocity. The trend of increasing enstrophy is nearly monotonic with increasing downflow. The enstrophy contrast between the strongest downflows and strongest updrafts is a factor of ten or more at most depths. The vertical component of enstrophy (dotted curves) is enhanced in downflows in the upper most pressure scale heights as well as in updrafts near the lower boundary.

Fig. A16.— Dependence of enstrophy production terms on filtered vertical velocity in the same five horizontal slices as in Figures 15a-15e. Solid, dashed, and dotted lines correspond to the vortex stretching, baroclinic, and compressional terms in the enstrophy equation (see eqns. (3.21) – (3.24)). The Log_{10} of these terms is plotted here. The vortex stretching term dominates everywhere. All three terms are positive, and enhance the enstrophy, in downflows. The mean compressional term is negative, and acts to diminish enstrophy, in updrafts (negative values are indicated with X's).

Fig. A17.— Dependence of divergence of velocity on filtered vertical velocity in the same five horizontal slices as in Figures 15a-15e. Solid, dashed, and dotted lines correspond to total divergence of velocity, divergence of velocity in the horizontal, and divergence of velocity in the vertical, respectively. As expected, the total divergence of velocity (solid curve) is positive in updrafts and negative in downdrafts. The divergence of velocity in the vertical (dashed line) is positive in updrafts along the bottom and in downflows along the top, which is consistent with the overall convective flow. Comparison of all five panels here with the corresponding ones in Figure 16 show that the direction of vorticity is aligned with the principal direction of large–scale strain everywhere.

Fig. A18.— Dependence of relative temperature fluctuations on filtered vertical velocity in the same five horizontal slices as in Figures 15a-15e. Solid lines show the mean value, and error bars show the *rms* fluctuations about the local mean, of the relative temperature fluctuations. As expected, updrafts are relatively warm, and downdrafts are relatively cool. The amplitude of temperature fluctuations is much smaller in updrafts than downdrafts, except along the lower boundary. Hence, thermal turbulence, like enstrophy, is enhanced in downflows and diminished in updrafts.