An Introduction to the Piecewise Parabolic Method (PPM) Library I. Interpolation and Hydrodynamics Routines: Part 4

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Figure Captions

Figure 1: These are comparisons of zone averages of a function to the function itself. In the top graph (figure 1a), the generating function is the sine wave depicted by the dotted line. Zone averages of this function on a uniform grid are shown as a solid line. In the bottom graph (figure 1b), the generating function a hyperbolic tangent, depicted by the dotted line. Zone averages of this function on a non-uniform grid ar shown as a solid line. These functions (or such similar) will be used to illustrate the interpolative routines of PPMLIB.

Figure 2: This is a sequence of zone average data sets (solid lines), and their generating functions (dotted lines) illustrating the transition from a well resolved distribution to a poorly resolved distribution. In this sequence of four sets, the width of the transition from -1 to +1 in the underlaying generating function decreases from approximately 8 zones to 1 zone by a factor of two between each set. All are created on a 20 zone non-uniform grid.

Figure 3: Results from a program to approximate the second derivative from zone averaged data. The top graph shows the initial zone average distribution. The bottom graph is the results from a program which calculated second derivatives.

Figure 4: This shows the results from PPMLIB interpolations based on zone-averaged data. The initial data are that of Figure 1a, a well resolved sine wave (50 zones per wavelength) shown as a dotted line. A piecewise parabolic interpolation constructed from zone averages of this function (not shown here) is shown as a dashed line.These lines are virtually coincident.

Figure 5: Here are compared piecewise parabolic interpolations from program `INTERPOLATE_2`, with the generating functions for the data sequence illustrated in figure 2. First, zone averages (not shown here) are generated from a hyperbolic tangent function on non-uniform grids (dotted lines). PPM interpolations (without monotonicity constraints or discontinuity detection) are computed and plotted as dashed lines. For smooth, well resolved transitions, the interpolations and generating functions are virtually identical. However, the small departures just visible in the 4 zone jump become much more pronounced as the width (measured in number of zones) of the jump narrows. The final two interpolations clearly illustrates "ringing," the spurious over and undershoots shown here. For many applications, this will lead eventually to over- and undershoots in the zone averaged data.

Figure 6: Here are compared piecewise parabolic interpolations, with monotonicity constraints, with the generating functions for the same data sequence illustrated in figure 5. The actual function used to

generate the zone averages is shown as a dotted line. The zone averages themselves are not shown. The monotonized interpolation, computed by INTERPOLATE_3, is shown as a dashed line. A comparison with figure 5 shows that in addition to preventing spurious oscillations, the application of monotonicity constraints also provides a better representation for unresolved jumps in the data.

Figure 7: Here are compared piecewise parabolic interpolations, with monotonicity constraints, with the generating functions for the same data sequence illustrated in figures 5 and 6. The actual function used to generate the zone averages is shown as a dotted line. The zone averages themselves are not shown. This shows how discontinuity detection produces an improved interpolation over the previous case. It should be noted that the discontinuities at the edges of the central zones do not violate monotonicity, as used in PPMLIB. It is in the context of advection that monotonicity is imposed, not on a point by point basis.

Figure 8: This figure illustrates the geometry for integrating over portions abutting the left and right hand zone interfaces of a grid zone.

Figure 9: This figure illustrates one procedure from remapping (regridding) data. Interpolations (solid line) determined one the old grid are integrated over regions of overlap (shaded regions) between the old grid and the new grid, creating zone interface fluxes. New zone averages may be found on the new grid by conservatively differencing these fluxes the geometry for integrating over portions abutting the left and right hand zone interfaces of a grid zone.

Figure 10: This shows the results of advected a sine waves resolved over 20 zones on a non-uniform grid. This initial zone average data distribution is shown as dotted lines. The plot on the left shows the results of advecting the distribution leftward by an amount equal to the smallest zone width. The plot on the right shows the results of advecting the distribution rightward by an amount equal to the smallest zone width.

Figure 11: This shows the results of advecting a data set data consisting of zone averages of a single period sine wave on a 20 zone uniform grid (unlike figure 10). section. The initial zone average distribution is shown as a solid line. The result of 2000 iterations (amounting to 10 revolutions) at a fractional step size of 0.1, is shown as dotted line. Even after 2000 interpolations and integrations, the distribution is little changed.

Figure 12: On the left are the results of 1200 iterations (60 revolutions at an offset of 0.5 zones per iteration) on a 10 zone sine wave. As this is near the interpolation resolution limit based on PPMLIB default parameters, it is not surprising to see that the waveform has undergone significant modification. The monotonicity constraints have clipped the extrema causing the sine wave to evolve towards a square wave. Numerical dissipation has also reduced the amplitude.

he graph on the right shows the same number of revolutions but with the PPMLIB parameters controlling the determination of function smoothness modified to

$$\Sigma_0 = 0.4$$

and

$$\Sigma_1 = 2.5$$

in an attempt to follow a 10 zone sine wave.
While the numerical dissipation is
still present, the wave form is relatively unscathed.

Figure 13: The effective states for the Riemann problems are found by finding the spatial averages over the *domains of dependence* for each zone interface. The plus domain of dependence may be approximated by tracing back the paths of all sound waves travelling in the positive direction with respect to the

fluid (that is, those waves travelling with a velocity of u+c in the laboratory frame of reference) which may impinge upon the zone interface from the zone to the left of the interface during the timestep. Likewise, the minus domain of dependence may be approximated by tracing back the paths of all sound waves travelling in the negative direction with respect to the fluid (that is, those waves travelling with a velocity of u-c in the laboratory frame of reference) which may impinge upon the zone interface from the zone to the right of the interface during the timestep. Finally, the streamline characteristic is traced backwards. It should be noted however, that this requires knowledge of the time averaged velocity at the zone interface, complicating the process somewhat. On the left is a depiction of the subsonic case, on the right is a supersonic case, with flow from the left.

Figure 14: The determination of effective states for a zone interface usually results in a discontinuity at that interface, which must now be resolved. The break up of an arbitrary fluid discontinuity is described by a solution to Riemann's shock tube problem. In hydrodynamics, this solution usually consists of three states, the effective left and right states, U_L and U_R, and a middle state U_*, separated by waves which may be shocks or rarefactions.

Figure 15: This is a typical computed solution to a one-dimension shock tube problem i(Sod 1978).

Figure 16: The initial values of the two passively advected diagnostics used in the tutorials. One is a Gaussian distribution and the other is a ''picket fence'' diagnostic consisting of alternating strips of ones and zeros. Either may be mass or volume weighted when advected.

Figure 17: A representative plot of the two diagnostics shown in figure 16 at the end of 100 timesteps is shown of the one-dimensional shocktube problem (figure 15) The initial discontinuity was at the center of the grid. This dramatically illustrates the ability of the steepening operation to maintain sharp features.

Figure 18: Here are the results from a ''two-dimensional shocktube,'' which is a rectangular computational domain divided into four smaller rectangles or subdomains, with all four meeting at the point near the center. Thus there is a discontinuity in both directions. Each subdomain consists of a constant, but different, hydrodynamic state. Images for four different times are shown here. Each image is a composite of density, total energy, x-velocity and y-velocity (left to right, top to bottom within each image). The time sequence is left to right, top to bottom. That is, the earliest image is in the top left, the latest in the bottom right. The configuration resembles a shaped charge, creating a jet of material from the highest pressure quadrant moving into the lowest pressure quadrant. Periodic boundary conditions are used, thus producing four such jets.

Figure 19: Initial state for a 3d shock tube. Under computation

Figure 20: Representative results from the problem depicted in figure 19. Under computation

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